SPRING 2008 FINAL EXAM

1. [24 Points] Evaluate the following integrals:

a) [12 Points] $(\ast) = \int \frac{9x-3}{(2x-1)(x+1)}\,dx \quad = \int \frac{A}{2x-1} + \frac{B}{x+1}\,dx$

$9x-3 = A(x+1) + B(2x-1)$ $\quad x=-1: \; -12 = A(0) + B(-3)$ so $B=4$

$x=\frac{1}{2} \quad \frac{9}{2}-3=\frac{3}{2} = A(\frac{3}{2}) + B0$ so $A=1$

So $(\ast) = \int \frac{1}{2x-1} + \frac{4}{x+1}\,dx$

$= \frac{1}{2}\ln|2x-1| + 4\ln|x+1| + C$

b) [12 Points] $(\ast) = \int x^4 \ln x\,dx$

$u=\ln x \quad dv = x^4 dx$

$du = \frac{1}{x}dx \quad v=\frac{1}{5}x^5$

$(\ast) = (\frac{1}{5}x^5)(\ln x) - \int (\frac{1}{5}x^5)(\frac{1}{x}dx) = \frac{1}{5}x^5\ln x - \frac{1}{5}\int x^4\,dx$

$= \frac{1}{5}x^5\ln x - \frac{1}{25}x^5 + C$

2. [12 Points] For what values of x does the following series converge? For these values of x, what is the sum of the series?

$$(\ast) = \sum_{k=1}^{\infty}(-1)^{k+1}(\frac{3}{5}x)^k = \frac{3}{5}x - (\frac{3}{5}x)^2 + (\frac{3}{5}x)^3 - \cdots \qquad (1)$$

Treat as a geometric series!

$(\ast) = \frac{3}{5}x\left(1 + (-\frac{3}{5}x) + (-\frac{3}{5}x)^2 + (-\frac{3}{5}x)^3 + \cdots\right)$

$= \frac{3}{5}x\sum_{n=0}^{\infty}(-\frac{3}{5}x)^n$ converges only for $|-\frac{3}{5}x|<1$, ie $|x| < 5/3$.

The sum is $\frac{3}{5}x\frac{1}{1-(-\frac{3}{5}x)} = \frac{3}{5}\frac{x}{1+\frac{3}{5}x} = \frac{3x}{5+3x}$

3. [18 Points] Determine whether the following improper integrals are convergent or divergent. If the integral is convergent, find its exact value.

a) [9 Points] $\int_1^\infty \frac{e^{-\sqrt{x}}}{\sqrt{x}}\,dx$

$u=\sqrt{x}$ $du = \frac{1}{2\sqrt{x}}dx$ $\frac{1}{\sqrt{x}}dx = 2du$

$x=1 \rightsquigarrow u=1$ $x=\infty \rightsquigarrow u=\infty$.

$$= \int_1^\infty 2e^{-u}du = \lim_{N\to\infty} -2e^{-u}\Big|_1^N = \lim_{N\to\infty} 2e^{-1} - 2e^{-N} = 2e^{-1} - 0$$

converges

b) [9 Points] $\int_0^1 \frac{x}{\sqrt{1-x^2}}\,dx$

$u = 1-x^2$ $du = -2x\,dx$ $x\,dx = -\frac{1}{2}du$

$x=0 \to u=1$ $x=1 \to u=0$

$$= \int_1^0 -\frac{1}{2}\frac{du}{\sqrt{u}} = \int_0^1 \frac{1}{2}u^{-1/2}du = u^{1/2}\Big|_0^1 = 1^{1/2} - 0^{1/2} = 1$$

converges.

4. [8 Points] By using a suitable comparison Theorem, determine whether the following improper integral is convergent or divergent:

$$(*) = \int_1^\infty \frac{x^2}{4x^3 - x + 1}\,dx$$

Limit comparison for improper integrals:

$\frac{x^2}{4x^3-x+1} = f(x)$, $g(x) = \frac{1}{x}$ then $\frac{f(x)}{g(x)} = \frac{x^3}{4x^3-x+1} = \frac{1}{4 - \frac{1}{x^2} + \frac{1}{x^3}}$

which $\to \frac{1}{4} \neq 0$ as $N\to\infty$, so either $(*)$ and $\int_1^\infty \frac{1}{x}dx$ both converge or both diverge. But

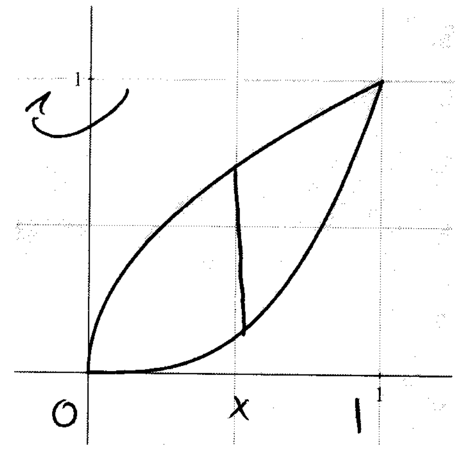
$\int_1^\infty \frac{1}{x}dx = \lim_{N\to\infty} \ln|x|\Big|_1^N = \lim_{N\to\infty} \ln N = \infty$ diverges, so

$(*)$ diverges.

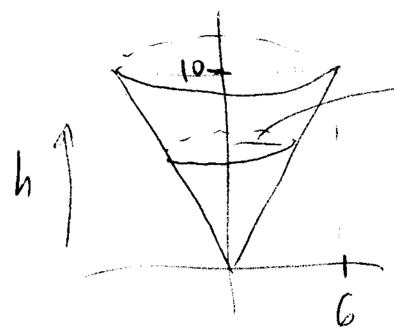
5. [12 Points] Let **R** be the bounded region in the first quadrant enclosed by the graphs of $y = x^3$ and $y = \sqrt{x}$, as shown. Let **S** be the solid obtained by revolving **R** about the **y-axis**. Find **(but don't evaluate)** an integral whose value gives the volume of **S**. You may use any method of your choice.

$$\text{Volume} = \int_{\text{left}}^{\text{right}} 2\pi (\text{radius})(\text{height})\, dx$$

$$= \int_0^1 2\pi (x-0)(\sqrt{x} - x^3)\, dx$$



6. [12 Points] A tank has the shape of a right circular cone with its vertex on the ground. The height of the tank is 10 feet; the radius of its top is 6 feet. Assume that the tank is filled with oil weighing 50 lb/ft^3. Write down **but do not evaluate** an integral whose value is the work required to pump all of the oil over the top of the tank.



$$\text{radius} = \frac{6}{10} h = \frac{3}{5} h$$

$$\text{work} = \int_{\text{bottom}}^{\text{top}} (\text{density})(\text{area of slice})(\text{distance lifted})\, dh$$

$$= \int_0^{10} (50)\left(\pi \left(\tfrac{3}{5} h\right)^2\right)(10-h)\, dh$$

7. [16 Points] Determine whether the following series converge absolutely, converge conditionally or diverge. You must show all details to receive credit.

a) [6 Points] $\sum_{k=1}^{\infty} \frac{k}{2k+3}$ $= \sum a_n$ $\lim a_n = \lim \frac{k}{2k+3} = \lim \frac{1}{2+3/k} = \frac{1}{2} \neq 0$

so diverges by n^{th} term test.

b) [10 Points] $\sum_{k=2}^{\infty} (-1)^k \frac{\sqrt{k}}{3k^2+k-1}$ $= \sum (-1)^k a_k$ alternating series

$a_k = \frac{\sqrt{k}}{3k^2+k-1}$ a_k decreasing (yes but tough to show?) $\rightarrow$ alt. series test will work.

or: $\sum | \, | = \sum \frac{\sqrt{k}}{3k^2+k-1}$ limit compare with $\sum \frac{\sqrt{k}}{3k^2} = \sum b_k$

$\frac{|(-1)^k a_k|}{b_k} = \frac{3k^2}{3k^2+k-1} \rightarrow 1 \neq 0$ as $k \rightarrow \infty$ so since

$\sum b_k = \frac{1}{3}\sum \frac{1}{k^{3/2}}$ converges (p-series, $p = \frac{3}{2} > 1$),

$\sum (-1)^k a_k$ converges.

8. [12 Points] A particle is traveling in space from the point $P = (2, 4, -2)$ to the point $Q = (6, 2, 2)$ on a line segment with speed $2\, cm/min$, where the xyz-coordinate system is measured in centimeters.

a) [6 Points] Find the velocity vector of the particle and total time needed for the trip.

direction $= \vec{PQ} = (4, -2, 4)$ want speed $= 2$

$\|(4,-2,4)\| = (16+4+16)^{1/2} = 36^{1/2} = 6$ so velocity is

$\cancel{\frac{2}{6}}\ 2 \cdot \frac{\vec{PQ}}{\|\vec{PQ}\|} = 2\,\frac{(4,-2,4)}{6} = \frac{1}{3}(4,-2,4) = \left(\frac{4}{3}, \frac{-2}{3}, \frac{4}{3}\right)$

b) [6 Points] Find the parametric equations for the path of the particle.

$\vec{r}(t) = P + t\vec{v} = \langle 2, 4, -2 \rangle + t\left\langle \frac{4}{3}, \frac{-2}{3}, \frac{4}{3} \right\rangle$

$= \left\langle 2 + \frac{4}{3}t, 4 - \frac{2}{3}t, -2 + \frac{4}{3}t \right\rangle$

9. [24 Points] Find the radius of convergence and the largest interval on which the following power series converges absolutely. At each endpoint of the interval, determine whether the series is absolutely convergent, conditionally convergent, or divergent.

$$\sum_{k=1}^{\infty} \frac{(-1)^k}{k}(x-2)^k = \sum a_n (x-2)^k \tag{2}$$

$$a_n = \frac{(-1)^k}{k}$$

$$\left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{\frac{1}{k+1}}{\frac{1}{k}}\right| = \frac{k}{k+1} = \frac{1}{1+1/k} \to 1 = L$$

so radius of convergence $= \frac{1}{L} = \frac{1}{1} = \underline{\underline{1}}$

so conv <u>absolutely</u> on $|x-2| < 1$ i.e., $1 < x < 3$

diverges on $|x-2| > 1$

Check endpoints! $x=1$ $\quad \sum \frac{(-1)^k}{k}(-1)^k = \sum \frac{1}{k}$ <u>diverges</u>

$x=3$ $\quad \sum \frac{(-1)^k}{k}(1)^k = \sum \frac{(-1)^k}{k}$ ~~d~~ converges

(alt. harmonic) $\frac{1}{k} \to 0$ as $k \to \infty$

but <u>not</u> <u>absolutely</u>.

so: <u>absolute</u> convergence on $1 < x < 3$

<u>convergence</u> on $1 < x \leq 3$

10. [16 Points] The Taylor series of e^{-x^2} about $x = 0$ is:

$$e^{-x^2} = \sum_{k=0}^{\infty} (-1)^k \frac{1}{k!}\, x^{2k}, \quad x \in \mathbb{R}. \tag{3}$$

a) [6 Points] Find a series whose sum is $\int_0^1 e^{-x^2}\,dx$.

$$\int e^{-x^2}dx = \int \sum (-1)^k \frac{1}{k!} x^{2k}\,dx = \sum (-1)^k \frac{1}{k!} \frac{x^{2k+1}}{2k+1}. \text{ So}$$

$$\int_0^1 e^{-x^2}dx = \sum (-1)^k \frac{1}{k!} \frac{1^{2k+1}}{2k+1} - \sum (-1)^k \frac{1}{k!}\frac{0^{2k+1}}{2k+1} \quad 0!$$

$$= \sum_{k=0}^{\infty} \frac{(-1)^k}{k!(2k+1)}$$

b) [10 Points] Use the series you found in part a) to approximate integral $\int_0^1 e^{-x^2}\,dx$ with an error that does not exceed 0.03.

This is an alternating series! $\sum (-1)\, a_k$.

Since a_k are decreasing, for

$L = \sum_{k=0}^{\infty} \frac{(-1)^k}{k!(2k+1)}$ and $S_N = \sum_{k=0}^{N} \frac{(-1)^k}{k!(2k+1)}$,

$|L - S_N|$ is smaller than $a_{N+1} = \frac{1}{N!(2N+1)}$. This is less than 0.03 provided $\frac{1}{N!(2N+1)} < .03$, i.e.

$\frac{100}{3} = \frac{1}{.03} < N!(2N+1)$ so check! $N=1$ $1!(2N+1)=3$ no

$N=2$ $2!(4+1)=10$ no

so $N+1=3$ works

$N=3$ $3!(6+1)=42 > \frac{100}{3}$

$$\text{so } L \approx \sum_{k=0}^{2} \frac{(-1)^k}{k!(2k+1)} = \frac{1}{0!\,1} - \frac{1}{1!\,3} + \frac{1}{2!\,5} = 1 - \frac{1}{3} + \frac{1}{10},$$

to within 0.03.

11. [20 Points] Consider the vectors $\overrightarrow{v} = \langle 3, -1, -1\rangle = 3\overrightarrow{i} - \overrightarrow{j} - \overrightarrow{k}$ and $\overrightarrow{w} = \langle 1, 4, -1\rangle = \overrightarrow{i} + 4\overrightarrow{j} - \overrightarrow{k}$.

a) [5 Points] Find the vector $\overrightarrow{v} + 2\overrightarrow{w}$.

$$\vec{v} + 2\vec{w} = \langle 3,-1,-1\rangle + 2\langle 1,4,-1\rangle$$
$$= \langle 3,-1,-1\rangle + \langle 2,8,-2\rangle$$
$$= \langle 5,7,-3\rangle$$

b) [5 Points] Find the cosine of the angle between $\overrightarrow{v}$ and $\overrightarrow{w}$.

$$\cos\theta = \frac{\vec{v}\cdot\vec{w}}{\|\vec{v}\|\|\vec{w}\|} = \frac{(3-4+1)}{(3^2+1^2+1^2)^{1/2}(1^2+4^2+1^2)^{1/2}} = \frac{0}{11^{1/2}\,18^{1/2}} = \underline{\underline{0}}$$

c) [5 Points] Find $\text{proj}_{\overrightarrow{w}}\overrightarrow{v}$, i.e., the vector projection of $\overrightarrow{v}$ onto $\overrightarrow{w}$.

$$\text{proj}_{\vec{w}}\vec{v} = \frac{\vec{v}\cdot\vec{w}}{\vec{w}\cdot\vec{w}}\vec{w} = \frac{0}{1^2+4^2+1^2}\langle 1,4,-1\rangle = 0\langle 1,4,-1\rangle$$
$$= \langle 0,0,0\rangle$$

d) [5 Points] Does there exist a **unit** vector $\overrightarrow{u}$ that is parallel to the vector $\overrightarrow{w}$ and orthogonal to $\overrightarrow{v}$? If "yes", find it; and if "no" explain why not.

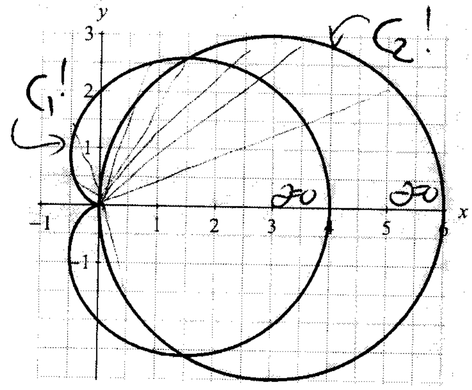
Yes! $\vec{w}$ is orthogonal to $\vec{v}$, so the unit vector in the direction $\vec{w}$, $\frac{\vec{w}}{\|\vec{w}\|} = \frac{\langle 1,4,-1\rangle}{\sqrt{18}} = \vec{u}$ has $\|\vec{u}\| = 1$ and $\vec{u} \perp \vec{v}$

12. [26 Points] The graphs of the following polar curves are as shown:

$$C_1: \quad r = 2(1+\cos\theta); \qquad C_2: \quad r = 6\cos\theta.$$

Points of intersection:

$$2(1+\cos\theta) = 6\cos\theta$$
$$2 + 2\cos\theta = 6\cos\theta$$
$$2 = 4\cos\theta \qquad \cos\theta = \frac{1}{2}$$
$$\theta = \frac{\pi}{3}, \ \theta = -\frac{\pi}{3}$$



a) [10 Points] Find, but don't evaluate, an integral whose value is the area of the region that lies outside C_1 and inside C_2. (You may use any available symmetry).

$$\text{Area} = \int_{\text{start}}^{\text{finish}} \frac{1}{2}\left((\text{outside})^2 - (\text{inside})^2 \right) d\theta$$

$$= \int_{-\frac{\pi}{3}}^{\frac{\pi}{3}} \frac{1}{2}\left((6\cos\theta)^2 - (2(1+\cos\theta))^2 \right) d\theta$$

b) [8 Points] Find the slope of C_2 at the point that corresponds to $\theta = \frac{\pi}{3}$.

$$r = 6\cos\theta \qquad x = 6\cos^2\theta \qquad y = 6\cos\theta\sin\theta$$

$$\frac{dx}{d\theta} = 12\cos\theta(-\sin\theta) \qquad \frac{dy}{d\theta} = 6\left((-\sin\theta)\sin\theta + (\cos\theta)(\cos\theta)\right)$$

At $\theta = \frac{\pi}{3}$, $\frac{dx}{d\theta} = 12\left(\frac{1}{2}\right)\left(-\frac{\sqrt{3}}{2}\right) = -3\sqrt{3}$, $\frac{dy}{d\theta} = 6\left(\left(-\frac{\sqrt{3}}{2}\right)\left(\frac{\sqrt{3}}{2}\right) + \left(\frac{1}{2}\right)\left(\frac{1}{2}\right)\right)$

$$= 6\left(-\frac{3}{4} + \frac{1}{4}\right) = 6\left(-\frac{1}{2}\right) = -3$$

So $\frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta} = \frac{(-3)}{-3\sqrt{3}} = \frac{1}{\sqrt{3}}$ reasonable?

Look at the picture at $\frac{\pi}{3} = \theta$!

c) [8 Points] Find, but don't evaluate, an integral whose value is the arc length of C_1.

C_1 one time around the curve is $0 \le \theta \le 2\pi$

$$\text{Length} = \int_{\text{start}}^{\text{finish}} \left((f'(\theta))^2 + (f(\theta))^2 \right)^{1/2} d\theta \qquad \begin{aligned} f(\theta) &= 2(1+\cos\theta) \\ f'(\theta) &= 2(-\sin\theta) \end{aligned}$$

$$= \int_0^{2\pi} \left((2(-\sin\theta))^2 + (2(1+\cos\theta))^2 \right)^{1/2} d\theta.$$