## Math 106 Exam 2 Topics

### The Chain Rule

Composition $(g \circ f)(x_0) = g(f(x_0))$ ; (note: we <u>don't</u> know what $g(x_0)$ is.)
$(g \circ f)'$ ought to have something to do with $g'(x)$ and $f'(x)$
in particular, $(g \circ f)'(x_0)$ should depend on $f'(x_0)$ and $g'(f(x_0))$

Chain Rule: $(g \circ f)'(x_0) = g'(f(x_0))f'(x_0)$
$= (d(\text{outside})$ eval'd at inside fcn$)\cdot(d(\text{inside}))$
Ex: $((x^3 + x - 1)^4)' = (4(x^3 + 1 - 1)^3)(3x^2 + 1)$

Different notation:
$y = g(f(x)) = g(u)$, where $u = f(x)$, then $\dfrac{dy}{dx} = \dfrac{dy}{du}\dfrac{du}{dx}$

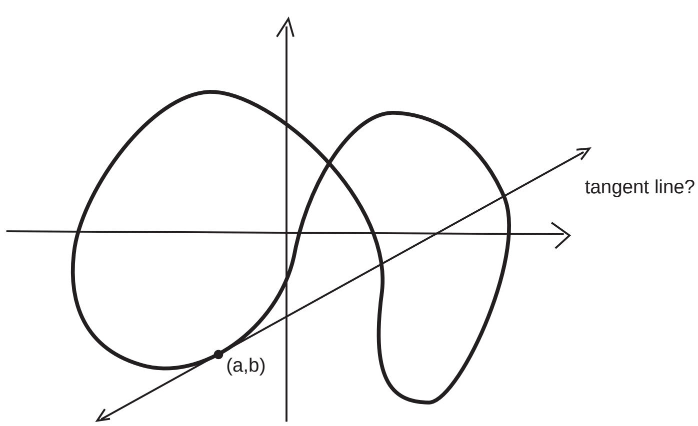
Parametric equations: a general curve needn't be the graph of a function. But we can imagine ourselves travelling along a curve, and then $x = x(t)$ and $y = y(t)$ are functions of $t$=time. We still may have a reasonable tangent line to the graph, and its slope should still be

(change in $y$/change in $x = \lim_{t\to t_0} \dfrac{y(t) - y(t_0)}{x(t) - x(t_0)} = \lim_{t\to t_0} \dfrac{(y(t) - y(t_0))/(t - t_0)}{(x(t) - x(t_0))/(t - t_0)}$

$= \dfrac{\lim_{t\to t_0}(y(t) - y(t_0))/(t - t_0)}{\lim_{t\to t_0}(x(t) - x(t_0))/(t - t_0)} = \dfrac{y'(t_0)}{x'(t_0)}$

### Implicit differentiation

We can differentiate functions; what about *equations*? (e.g., $x^2 + y^2 = 1$)
graph <u>looks</u> <u>like</u> it has tangent lines




Idea: <u>Pretend</u> equation defines $y$ as a function of $x$ : $x^2 + (f(x))^2 = 1$ and differentiate!
$2x + 2f(x)f'(x) = 0$ ; so $f'(x) = \dfrac{-x}{f(x)} = \dfrac{-x}{y}$

Different notation:
$x^2 + xy^2 - y^3 = 6$ ; then $2x + (y^2 + x(2y\dfrac{dy}{dx}) - 3y^2\dfrac{dy}{dx} = 0$
$\dfrac{dy}{dx} = \dfrac{-2x - y^2}{2xy - 3y^2}$

Application: extend the power rule

$\frac{d}{dx}(x^r) = rx^{r-1}$ works for any *rational* number $r$

($y = x^{p/q}$ means $y^q = x^p$ ; differentiate!)

**Inverse functions and their derivatives**

Basic idea: run a function backwards

y=f(x) ; 'assign' the value x to the input y ; x=g(y)

need g a function; so need f is one-to-one

f is one-to-one: if f(x)=f(y) then x=y ; if $x \neq y$ then $f(x) \neq f(y)$

$g = f^{-1}$, then $g(f(x)) = x$ and $f(g(x)) = x$ (i.e., $g \circ f$=Id and $f \circ g$=Id)

finding inverses: rewrite y=f(x) as x=some expression in y

graphs: if (a,b) on graph of $f$, then (b,a) on graph of $f^{-1}$

graph of $f^{-1}$ is graph of $f$, reflected across line y=x

horizontal lines go to vertical lines; horizontal line test for inverse

derivative of the inverse: $f'(f^{-1}(x)) \cdot (f^{-1})'(x) = 1$

if $f(a) = b$, then $(f^{-1})'(b) = 1/f'(a)$

**Logarithms**

$f(x)$=$a^x$ is **either** always increasing ($a > 1$) **or** always decreasing ($a < 1$)

inverse is $g(x) = \log_a x = \frac{\ln x}{\ln a}$

$\ln x$ is the inverse of $e^x$.

$\ln x$ is a log; it turns products into sums: $\ln(ab) = \ln(a) + \ln(b)$

$\ln(a^b) = b\ln(a)$ ; $\ln(a/b) = \ln(a) - \ln(b)$

$e^{\ln x} = x$ and $(e^x)' = e^x$ , so $1 = (e^{\ln x})' = (e^{\ln x})(\ln x)' = x(\ln x)'$, so $(\ln x)' = 1/x$ .

$\frac{d}{dx}(\ln x) = 1/\text{x}$ ; $\frac{d}{dx}(\ln(f(x))) = \frac{f'(x)}{f(x)}$ This gives us:

**Logarithmic differentiation:** $f'(x) = f(x)\frac{d}{dx}(\ln(f(x)))$

useful for taking the derivative of products, powers, and quotients

$\ln(a^b)$ **should be** $b\ln a$, so $a^b = e^{b\ln a}$

$a^{b+c} = a^b a^c$ ; $a^{bc} = (a^b)^c$

$a^x = e^{x\ln a}$ ; $\frac{d}{dx}(a^x) = a^x \ln a$

$x^r = e^{r\ln x}$ (makes sense for *any real number* $r$) ; $\frac{d}{dx}(x^r) = e^{r\ln x}(r)(\frac{1}{x}) = rx^{r-1}$

**Inverse trigonometric functions**

Trig functions ($\sin x$, $\cos x$, $\tan x$, etc.) aren't one-to-one; make them!

$\sin x$, $-\pi/2 \leq x \leq \pi/2$ is one-to-one; inverse is Arcsin $x$

sin(Arcsin $x$)=$x$, all $x$; Arcsin($\sin x$)=$x$ IF $x$ in range above

$\tan x$, $-\pi/2 < x < \pi/2$ is one-to-one; inverse is Arctan $x$

tan(Arctan $x$)=$x$, all $x$; Arctan($\tan x$)=$x$ IF $x$ in range above

$\sec x$, $0 \le x < \pi/2$ and $\pi/2 < x \le \pi$, is one-to-one; inverse is Arcsec $x$
  sec(Arcsec$x$)=$x$, all $x$; Arcsec($\sec x$)=$x$ IF $x$ in range above

Computing cos(Arcsin $x$), tan(Arcsec $x$), etc.; use right triangles

The other inverse trig functions aren't very useful,
  they are essentially the negatives of the functions above.

**Derivatives of inverse trig functions**

They are the derivatives of inverse functions! Use right triangles to simplify.

$$\frac{d}{dx}(\arcsin x) = \frac{1}{\cos(\arcsin(x))} = \frac{1}{\sqrt{1-x^2}}$$

$$\frac{d}{dx}(\arctan x) = \frac{1}{\sec^2(\arctan x)} = \frac{1}{x^2+1}$$

$$\frac{d}{dx}(\text{arcsec } x) = \frac{1}{\sec(\text{arcsec } x)\tan(\text{arcsec } x)} = \frac{1}{|x|\sqrt{x^2-1}}$$

**Related Rates**

Idea: If two (or more) quantities are related (a change in one value means a change in others), then their rates of change are related, too.

$xyz = 3$ ; pretend each is a function of $t$, and differentiate (implicitly).

General procedure:
Draw a picture, describing the situation; label things with variables.
Which variables, rates of change do you know, or want to know?
Find an equation relating the variables whose *rates of change* you know or want to know.
Differentiate!
Plug in the values that you know.

**Linear approximation and differentials**

Idea: The tangent line to a graph of a function makes a good approximation to the function, near the point of tangency.
Tangent line to $y = f(x)$ at $(x_0, f(x_0)$ : $L(x) = f(x_0) + f'(x_0)(x - x_0)$
$f(x) \approx L(x)$ for $x$ near $x_0$
Ex.: $\sqrt{27} \approx 5 + \frac{1}{2\cdot 5}(27-25)$, using $f(x) = \sqrt{x}$
$(1+x)^k \approx 1 + kx$, using $x_0$=0
$\Delta f = f(x_0 + \Delta x) - f(x_0)$, then $f(x_0 + \Delta x) \approx L(x_0 + \Delta x)$ translates to
$\Delta f \approx f'(x_0)\cdot\Delta x$
differential notation: $df = f'(x_0)dx$
So $\Delta f \approx df$, when $\delta x = dx$ is small
In fact, $\Delta f - df$ = (diffrnce quot $-f'(x_0))\Delta x$ = (small)$\cdot$(small) = really small, goes like $(\Delta x)^2$

**Chapter 4:** Applications of Derivatives

**Extreme Values**

$c$ is an (absolute) maximum for a function $f(x)$ if $f(c) \geq f(x)$ for every other $x$
$d$ is an (absolute) minimum for a function $f(x)$ if $f(d) \leq f(x)$ for every other $x$
max or min = extremum

Extreme Value Theorem: If $f$ is a continuous function defined on a closed interval $[a, b]$, then $f$ actually *has* a max and a min.

Goal: figure out where they *are*!
$c$ is a relative max (or min) if $f(c)$ is $\geq f(x)$ (or $\leq f(x)$) for every $x$ *near* $c$. Rel max or min = rel extremum.
An absolute extremum is either a rel extremum or an endpoint of the interval.

$c$ is a critical point if $f'(c) = 0$ or does not exist.
A rel extremum is a critical point.

**So** absolute extrema occur either at critical points *or* at the endpoints.
So to find the abs max or min of a function $f$ on an interval $[a, b]$ :
(1) Take derivative, find the critical points.
(2) Evaluate $f$ at each critical point and endpoint.
(3) Biggest value is maximum value, smallest is minimum value.

## The Mean Value Theorem

You can (almost) recreate a function by knowing its derivative

Mean Value Theorem: if $f$ is continuous on $[a, b]$ and differentiable on $(a, b)$, then there is at least one $c$ in $(a, b)$ so that

$$f'(c) = \frac{f(b) - f(a)}{b - a}$$

Consequences:
Rolle's Theorem: $f(a) = f(b) = 0$; between two roots there is a critical point.
So: If a function has no critical points, it has at *most* one root!
A function with $f'(x)$=0 is constant.
Functions with the same derivative (on an interval) differ by a constant.

## The First Derivative Test

$f$ is *increasing* on an interval if $x > y$ implies $f(x) > f(y)$
$f$ is *decreasing* on an interval if $x > y$ implies $f(x) < f(y)$
If $f'(x) > 0$ on an interval, then $f$ is increasing
If $f'(x) < 0$ on an interval, then $f$ is decreasing

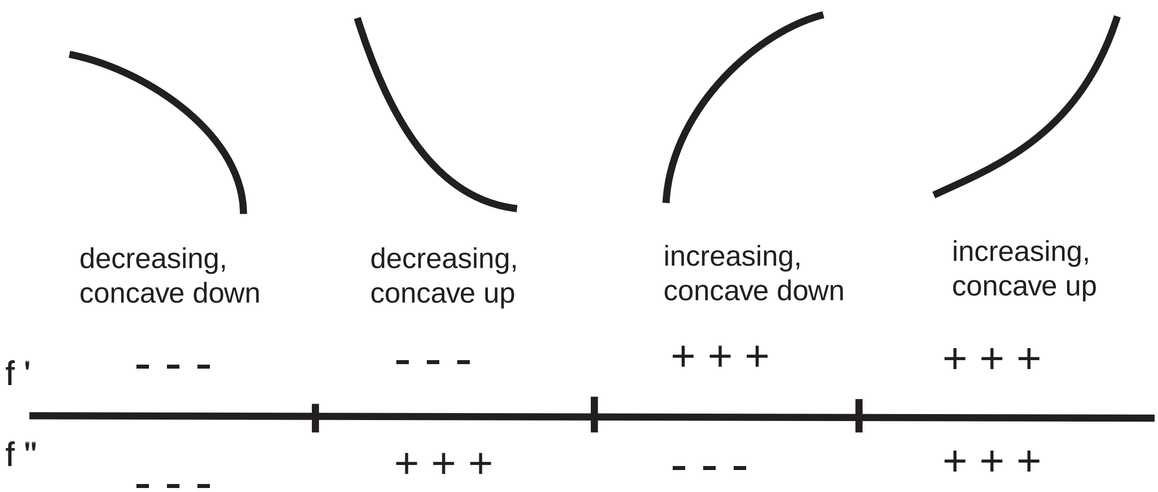
Local max's / min's occur at critical points; how do you tell them apart?
Near a local max, $f$ is increasing, then decreasing; $f'(x) > 0$ to the left of the critical point, and $f'(x) < 0$ to the right.
Near a local min, the opposite is true; $f'(x) < 0$ to the left of the critical point, and $f'(x) > 0$ to the right.
If the derivative does *not* change sign as you cross a critical point, then the critical point is not a rel extremum.
Basic use: plot where a function is increasing/decreasing: plot critical points; in between them, sign of derivative does not change.

**The second derivative test and graphing**

When we look at a graph, we see where function is increasing/decreasing. We also see:

$f$ is concave up on an interval if $f''(x) > 0$ on the interval
Means: $f'$ is increasing; $f$ is *bending* up.
$f$ is concave down on an interval if $f''(x) < 0$ on the interval
Means: $f'$ is decreasing; $f$ is *bending* down.
A point where the concavity changes is called a point of inflection

**Graphing:**
Find where $f'(x)$ and $f''(x)$ are 0 or DNE
Plot on the same line.
In between points, derivative and second derivative don't change sign, so graph looks like one of:




Then string together the pieces!
Use information about vertical and horizontal asymptotes to finish sketching the graph.

Second derivative test: If $c$ is a critical point and
$f''(c) > 0$, then $c$ is a rel min (smiling!)
$f''(c) < 0$, then $c$ is a rel max (frowning!)