Math 1650
Topics for third exam

(Technically, everything covered on the first and second exams, plus...)

Chapter 4: Trigonometry

§1: Degrees and radians
angle: vertex, initial side, terminal side
standard position: vertex=origin, initial side=(positive) x-axis
coterminal angles: same terminal side
measuring size of an angle
one full circle = 360 degrees
one full circle = 2π radians
radian measure = length of arc in circle of radius 1 swept out by the angle
acute, obtuse, reflex angles
$A+B = \pi/2$; complementary angles (acute)
$A+B = \pi$; supplementary angles (acute,obtuse)

§2: Trigonometric functions
In standard form, terminal side of an angle (t) meets circle of radius 1 in a point (x, y)
$x = \cos t$ = cosine of t
$y = \sin t$ = sine of t
$\frac{1}{x} = \frac{1}{\cos t} = \sec t$ = secant of t $\qquad$ $\frac{1}{y} = \frac{1}{\sin t} = \csc t$ = cosecant of t
$\frac{y}{y} = \frac{\sin t}{\cos t} = \tan t$ = tangent of t $\qquad$ $\frac{x}{y} = \frac{\cos t}{\sin t} = \cot t$ = cotangent of t
Examples:
$\sin(\pi/4) = \cos(\pi/4) = \sqrt{2}/2$
$\sin(\pi/6) = 1/2$; $\cos(\pi/6) = \sqrt{3}/2$
$\sin(\pi/3) = \sqrt{3}/2$; $\cos(\pi/3) = 1/2$
$\sin(\pi/2) = 1$; $\cos(\pi/2) = 0$ $\qquad$ $\sin(0) = 0$; $\cos(0) = 1$
Domain of $\sin t$, $\cos t$: all t
Range: $[-1, 1]$
point on circle corresp. to $t + 2\pi$ is same as point for t
$\sin(t + 2\pi) = \sin t$; $\cos(t + 2\pi) = \cos t$
$\sin t$ and $\cos t$ are periodic
symmetry:
$\cos t$, $\sec t$ are even functions
$\sin t$, $\csc t$, $\tan t$, $\cot t$ are odd functions

$x^2 + y^2 = 1$ (unit corcle): $\sin^2 t + \cos^2 t = 1$

§3: Right angle trigonometry
Right triangle:
$\sin(\theta) = a/c$ = (opposite)/(hypotenuse)
$\cos(\theta) = b/c$ = (adjacent)/(hypotenuse)
$\tan(\theta) = a/b$ = (opposite)/(adjacent)
"SOHCAHTOA"
Copmplementary angle = the 'other' angle in a right triangle
$\sin(\pi/2 - \theta) = \cos(\theta)$, $\qquad$ $\cos(\pi/2 - \theta) = \sin(\theta)$
$\tan(\pi/2 - \theta) = \cot(\theta)$, $\qquad$ $\cot(\pi/2 - \theta) = \tan(\theta)$
$\sec(\pi/2 - \theta) = \csc(\theta)$, $\qquad$ $\csc(\pi/2 - \theta) = \sec(\theta)$
(i.e., function("co-angle") = "co-function"(angle))

§4: Trig functions for any angle

Right angle trig really applies only to **acute** angles; extend the ideas!

angle θ, point (x, y) on terminal side

$r = \sqrt{x^2 + y^2}$

$\sin(\theta) = y/r \qquad \cos(\theta) = x/r \qquad \tan(\theta) = y/x$

reference angle = **acute** angle that terminal side makes with x-axis

(trig fcn)(θ) = (trig fcn)(ref. angle), **except** possibly for a change in sign:

II	I
$(x < 0, y > 0)$	$(x > 0, y > 0)$
$\sin(\theta) > 0$ $\cos(\theta) < 0$ $\tan(\theta) < 0$	$\sin(\theta) > 0$ $\cos(\theta) > 0$ $\tan(\theta) > 0$
$\sin(\theta) < 0$ $\cos(\theta) < 0$ $\tan(\theta) > 0$	$\sin(\theta) < 0$ $\cos(\theta) > 0$ $\tan(\theta) < 0$
$(x < 0, y < 0)$	$(x > 0, y < 0)$
III	IV

§5: Graphs of sine, cosine

$\sin(\theta)$ = y-value of the points (counter-clockwise) on the unit circle, starting with 0

$\cos(\theta)$ = x-value of the points (counter-clockwise) on the unit circle, starting with 1

Graph: note x-intercepts, y-intercept, maximum and minimum; draw a smooth curve

Transformations: $y = a\sin(bx)$

vertical stretch by factor of a; **amplitude** is $|a|$

amplitude = how far trig function wanders from its 'center'

horizontal compression by factor of b; **period** is $2\pi/|b|$

Translations: just like before

$y = \cos(x - a)$; translation to right by a

$y = \cos(x) + a$; translation up by a

§6: Graphs of other trig functions

$\tan x$, $\cot x$, $\sec x$ $\csc x$

Transformations (same)
Products: $\sin x$, $\cos x$ bounce between -1 and 1; so, for example:
$y = x \sin x$ bounces between $y = x$ and $y = -x$
$y = e^{-x} \cos x$ bounces between $y = e^{-x}$ and $y = -e^{-x}$ ('damped' trig function)

§7: Inverse trig functions
Inverses of trig functions? No! Not one-to-one. Solution: *make* them one-to-one!
$f(x) = \sin x$, $-\pi/2 \leq x \leq \pi/2$, is one-to-one
inverse is called $\arcsin x$ = angle (between $-\pi/2$ and $\pi/2$) whose sine is x
$\sin(\arcsin x) = x$; $\arcsin(\sin x) = x$ **if** x is between $-\pi/2$ and $\pi/2$
$f(x) = \cos x$, $0 \leq x \leq \pi$, is one-to-one
inverse is called $\arccos x$ = angle (between 0 and π) whose cosine is x
$\cos(\arccos x) = x$; $\arccos(\cos x) = x$ **if** x is between 0 and π
$f(x) = \tan x$, $-\pi/2 < x < \pi/2$, is one-to-one
inverse is called $\arctan x$ = angle (between $-\pi/2$ and $\pi/2$) whose tangent is x
$\tan(\arctan x) = x$; $\arctan(\tan x) = x$ **if** x is between $-\pi/2$ and $\pi/2$
Graphs: take appropriate piece fo trig function, and flip it across the line $y = x$
$\cos(\arcsin x)$ = (cosine of angle whose **sine** is x) = $\sqrt{1-x^2}$; etc.

Chapter 5: Analytic trigonometry

§1: Using fundamental identities
Fundamental identities:
Reciprocal: $\csc x = \dfrac{1}{\sin x}$ $\qquad \sec x = \dfrac{1}{\cos x}$ $\qquad \cot x = \dfrac{1}{\tan x}$
Quotient: $\tan x = \dfrac{\sin x}{\cos x}$ $\qquad \cot x = \dfrac{\cos x}{\sin x}$
Pythagorean: $\sin^2 x + \cos^2 x = 1$ $\qquad \tan^2 x + 1 = \sec^2 x$ $\qquad \cot^2 x + 1 = \csc^2 x$
Complementarity: $\sin(\pi/2 - x) = \cos(x)$ $\qquad \tan(\pi/2 - x) = \cot(x)$ $\qquad \sec(\pi/2 - x) = \csc(x)$
$\cos(\pi/2 - x) = \sin(x)$ $\qquad \cot(\pi/2 - x) = \tan(x)$ $\qquad \csc(\pi/2 - x) = \sec(x)$
Symmetry: $\cos(-x) = \cos x$ $\qquad \sec(-x) = \sec x$
$\sin(-x) = -\sin x$ $\qquad \csc(-x) = -\csc x$ $\qquad \tan(-x) = -\tan x$ $\cot(-x) = -\cot x$
Trig substitution: rewrite expression in x by 'pretending' x=trig function
$\sqrt{a^2 - x^2}$; write $x = a \sin\theta$, then $\sqrt{a^2 - x^2} = a\cos\theta$
$\sqrt{a^2 + x^2}$; write $x = a \tan\theta$, then $\sqrt{a^2 + x^2} = a\sec\theta$
$\sqrt{x^2 - a^2}$; write $x = a \sec\theta$, then $\sqrt{x^2 - a^2} = \pm a\tan\theta$

§2: Checking trig identities
Basic differences: an identity is supposed to be true for **every** value of x;
an equation is **solved** for the correct values of x
Basic idea: use identities that we already **know** (like the list above)
convert things to sines and cosines
play with the two sides of the identity
add 0 ! multply and divide by the same expression!
Examples: $\csc x - \sin x = \dfrac{1}{\sec x \tan x}$

$$\frac{\tan x + \tan y}{1 - \tan x \tan y} = \frac{\cot x + \cot y}{\cot x \cot y - 1}$$

§3: Solving trig equations
Idea: just like exponential and logarithmic equations; try to rewrite as
(single trig function) = (single value)
Wrinkles:

Polynomials: $2\cos^2 x + 3\cos x + 1 = 0$; $(2\cos x + 1)(\cos x + 1) = 0$

$2\cos x + 1 = 0$ **or** $\cos x + 1 = 0$

Trig identities: $\tan x + \sec x{=}4$; $\tan x = 4 - \sec x$; square both sides

$tan^2x\ (= \sec^2 x - 1) = 16 - 8\sec x + \sec^2 x = ...$

Problem: 'ghost solutions' = solutions which 'appear' only after manipulating equation

(stupid) Ex: $\sin x = 1$ and $(\sin x)^2 = 1$ have different sets of solutions!

§4: Angle sum and difference formulas

$\sin(A+B) = \sin A\cos B + \cos A\sin B$
$\sin(A-B) = \sin A\cos B - \cos A\sin B$
$\cos(A+B) = \cos A\cos B - \sin A\sin B$
$\cos(A-B) = \cos A\cos B + \sin A\sin B$

Note: it is easy to derive any threee formulas from the remaining one, using even/odd and complementarity formulas.

$$\tan(A+B) = \frac{\sin(A+B)}{\cos(A+B)} = \frac{\tan A + \tan B}{1 - \tan A\tan B}$$
$$\tan(A-B) = \frac{\sin(A-B)}{\cos(A-B)} = \frac{\tan A - \tan B}{1 + \tan A\tan B}$$

Some uses: complex multiplication! (side trip to part of Section 6.5)

$(a+bi)(c+di) = (ac-bd) + (ad+bc)i$

pretend $z{=}a+bi{=}\cos A + i\sin A$, $z'{=}c+di{=}\cos B + i\sin B$, then this reads

$z\cdot z'{=}(\cos A\cos B - \sin A\sin B) + (\sin A\cos B + \cos A\sin B)i$

$=\cos(A+B) = i\sin(A+B)$

Problem: $z{=}a+bi{=}\cos A + i\sin A$. then $a^2+b^2{=}\sin^2 A + \cos^2 A{=}1$ (every time)

Solution: think $z{=}a+bi{=}r(\cos A + i\sin A)$, where

$r^2{=}a^2+b^2$;. i.e, think $z \leftrightarrow (a,b)$ (in plane) = point in plane at distance r from origin, making angle A with (positive) x-axis

i.e., think $z{=}a+bi \leftrightarrow (a,b) \leftrightarrow$ (distance,angle) ; **polar coordinates**

then complex multiplication **multiplies** distance and **adds** angles:

$(r(\cos A + i\sin A))(r'(\cos B + i\sin B)) = (rr')(\cos(A+B) + i\sin(A+B))$

Another use: find values of trig functions at new angles:

Example: $105° = 60° + 45°$ (i.e. $7\pi/12 = \pi/3{+}\pi/4$), so

$\cos(7\pi/12) = \cos(\pi/3+\pi/4) = \cos(\pi/3)\cos(\pi/4) - \sin(\pi/3)\sin(\pi/4) =$
$(1/2)(\sqrt{2}/2) - (\sqrt{3}/2)(\sqrt{2}/2) = (\sqrt{2}-\sqrt{6})/4$

§5: Multiple angle, product-to-sum formulas

Double angle formulas: set $A = B$ in formulas above!

$\sin(2A) = \sin(A+A) = 2\sin A\cos A$

$\cos(2A) = \cos(A+A) = \cos^2 A - \sin^2 a = 2\cos^2 a - 1 = 1 - 2\sin^2 A$

Triple angle? $\sin(3A){=}\sin(2A+A){=}....$

$\sin^2 x{=}(1-\cos(2x))/2$, $\quad \cos^2 x{=}(1+\cos(2x))/2$; these give

Half-angle formulas:

$\sin(x/2) = \sqrt{(1-\cos x)/2}$; $\quad \cos(x/2) = \sqrt{(1+\cos x)/2}$

$$\tan(x/2) = \frac{\sin x}{1+\cos x} = \frac{1-\cos x}{\sin x}$$

Product-to-sum formulas:

$\sin(A+B) + \sin(A-B) = 2\sin A\cos B$, so

$\sin A\cos B{=}\frac{1}{2}(\sin(A+B) + \sin(A-B))$ Simlarly,

$\cos A \cos B = \frac{1}{2}(\cos(A+B) + \cos(A-B))$, and

$\sin A \sin B = \frac{1}{2}(\cos(A-B) - \cos(A+B))$

Sum-to-product formulas:

set $A+B=x$, $A-B=y$ (solve: $A=\frac{x+y}{2}$, $B=\frac{x-y}{2}$), plug in above!

$$\sin x + \sin y = 2\sin\frac{x+y}{2}\cos\frac{x-y}{2}$$

$$\cos x + \cos y = 2\cos\frac{x+y}{2}\cos\frac{x-y}{2}$$

$$\cos x - \cos y = 2\sin\frac{x+y}{2}\sin\frac{x-y}{2}$$

OK, so what's the point? It's alot easier to remember what these formulas (in the previous two sections) **say** if you remember where they **come from**. We built all of these formulas up from **one formula**; $\cos(A-B) =$..... . If you remember how each follows one from the other, then you don't 'have to' remember the formula!