Math 189H Joy of Numbers Activity Log

Thursday, September 8, 2011

Niels Bohr: *"We all agree that your theory is crazy, but is it crazy enough?"*

Isaac Asimov: *"The most exciting phrase to hear in science, the one that heralds the most discoveries, is not 'Eureka!' (I found it!) but 'That's funny…' "*

Picking up where we had lest off, adding together multiples of the numbers 5 and 7, we could create any number greater than 23 (and some sporadic numbers smaller than that). The question we are exploring has been described by some as the 'Money Problem': in a country whose currency has only denominations of 5 units and 7 units, what size transactions can two people, 'A' (='Alice') and 'B' (='Bob') carry out? What we have found is that Alice can, by some combinations of 5's and 7's, give Bob any amount greater than 23. But what if we let Bob 'make change', i.e., give some combination of denominations back to Alice? If A gives B the combination $5x+7y$ of 5 and 7, and B gives to A the combination $5z+7w$, then the net exchange (from A to B) is

$$(5x + 7y) - (5z + 7w) = 5(x - z) + 7(y - w) = 5a + 7b$$

where $a = x-z$ and/or $b = y-w$ can now be negative (but are both still integers). In our current situation, we could fairly quickly see that since, e.g., we can make both 24 and 25 (as $2 \cdot 5 + 2 \cdot 7$ and $5 \cdot 5$, respectively), we can make 1 (as, subtracting, $3 \cdot 5 + (-2) \cdot 7$). and, we further realized, once we can carry out a transaction of 1 unit, we can carry out 2 by doubling the amounts above, and 3 by tripling (trebling?) them, and, in general, an exchance of n units can be done as $(3n) \cdot 5 + (-2n) \cdot 7$, i.e., A give B $3n$ 5's and B gives A $2n$ 7's.

If you can't imagine a country whose currency is in 5's and 7's (I don't think there is any), think about making change from a cash register that has run out of everything except dimes and quarters. What kinds of change can you give using only 10's and 25's?

Back to combinations! Flush with our success, we experimented with other denomination combinations. With 3 and 11 (OK, the only made-up currency name your instructor could still remember 25 hours later was 'squill'), after writing down lots of combinations, we discovered that $21 = 7 \cdot 3$ and $22 = 2 \cdot 11$ were one apart, so again we could transact 1 squill, and therefore any number of squill. If our currency came in denominations of 27 and 99, again after writing down lots of combinations (eased by your instructor's rather late-in-the-game 'discovery' that it is easier to repeatedly add 99 (as 100-1) than to add 27), we noted that $4 \cdot 27 = 108$ and $99 = 1 \cdot 99$ differ by 9, so we can transact 9 squill and therefore any multiple of 9 squill. We spent some effort trying to figure out how to build a combination giving us a number less than 9, without any success. This prompted us to try to figure out why we were failing to do better (i.e., smaller). Finally, we realized that both 27 and 99 are multiples of 9 (!), and we concluded that any combination of the two would also be a multiple of 9 (so nothing between 0 and 9 could be built as a combination; if 9 divides a (non-zero) number, then 9 cannot be larger than the number!). In symbols what we are saying is that since $27 = 9 \cdot 3$ and $99 = 9 \cdot 11$, then $27a + 99b = 9 \cdot (3a + 11b)$ is a multple of 9. So this means that 9 is the smallest positive number we can write as

$27a + 99b$. And, just as important, it says that all combinations of 27 and 99 must be multiples of 9. Since we can transact multiples of 9, this means that the transactions we can carry out are precisely the multiples of 9.

We continued with several more examples, but focused more on the question of 'What is the smallest (positive) value we can build as a combination of our two numbers?'. With (oh, dear, can I remember?) 201 and 704, what we tried to do was to find numbers built as combinations that were 'close together'; their difference would then perhaps make a smaller number than any we had before. Three 201's and one 704 made 603 and 704, with a difference of 101; and four 201's and one 704 made 804 and 704, with a difference of 100. But these differences have a difference of 1 (!), and so, if we work it out, we can write 1 as a combination of 201 and 704:

$101 = 1 \cdot 704 - 3 \cdot 201$ and $100 = 4 \cdot 201 - 1 \cdot 704$, so
$1 = 101 - 100 = (1 \cdot 704 - 3 \cdot 201) - (4 \cdot 201 - 1 \cdot 704) = 2 \cdot 704 - 7 \cdot 201$

and with that, we can make any number as a combination of 201 and 704. With 129 and 444, we struck upon the idea, similar to what we just did, of taking multiples of 129 (0 (!), 129, 258, 387, 516) until we got 'close' to 444, then subtracting to get a smaller number to play the exact same game with. So since 387 and 444 can both be expressed as combinations, their difference, 444-387=57 (no, these were not the numbers we were working with!) can be expressed as a combination. Then we started again with multiples of, essentially, the smallest number we'd created so far, until we got close to some number we had already built. So looking at 57, 114, 171, 228, 285, 342, 399, and 456, we find that 456-444=12, and 129-114=15. So we can build both 12 and 15, so we can build 15-12=3.

If we were persistent enough, we could then start writing out all of the multiples of 3; if we did that, we would then discover that every number we have written down starting from 129 and 444 (including 129 and 444) are multiples of 3 (!). In the end, what really happened was that we noticed without all of that work that $129 = 3 \cdot 43$ [OK, we would have, if these had been the numbers we were working with!] and $444 = 3 \cdot 148$. So, as before, everything we might build as $129a + 444b$ must be a multiple of 3, and we can build 3, so we can build every multiple of 3. So the numbers expressable as $129a + 444b$ for integers $a, b \in \mathbb{Z}$ are precisely the multiples of 3.

So what did we just do? In trying to understand what numbers can be expressed as a combination of integers m and n, we were led to try to find 'small' such numbers (knowing that any multiple of those numbers could be expressed as $ma + nb$ as well). What we found out was that, in all the case we tried, the smallest (positive) number d that we could discover would turn out to be a factor of both m and n, and that, since any $ma+nb$ would also be a multiple of d, the answer to our problem would be "the numbers $ma+nb$ consist of precisely the multiples of d". More than that, we struck upon a plan for how to find the 'smallest' d; (repeatedly!) take a 'small' number u that we have found, and subtract a multiple su of it from a number Q we've also found, if the multiple su is 'close' to Q (i.e., $Q - su$ is smaller than u !). The only question left to answer is: will this always work? Is the smallest number expressable as $ma + nb$ always a factor of both m and n? And will our subtraction plan always find the smallest combination possible? We will find out!