Math 189H Joy of Numbers Activity Log

Tuesday, October 4, 2011

Gil Kalai: "Counting pairs is the oldest trick in combinatorics... Every time we count pairs, we learn something from it."

Dorothy Parker: "The cure for boredom is curiosity. There is no cure for curiosity."

There are exactly 55 collections of numbers a, b, c, d so that every **positive** integer can be expressed as $ax^2 + by^2 + cz^2 + dw^2$ for some integers x, y, z, w. (Ramanujan, 1920s?). [$1, 1, 1, 1$ is one such collection (Lagrange, 1770s?).]

We started with further speculation, extending our observations from last time. We had seen that so long as n was not a multiple of 2 or 5, we were always able to find a k so that $10^k \underset{n}{\equiv} 1$. More than that, sometimes the smallest such k was $n-1$, and looking further, even when it wasn't smallest, sometimes $n-1$ 'worked', that is, $10^{n-1} \underset{n}{\equiv} 1$. When we looked still deeper, we found that $n-1$ always worked when n was prime, and $n-1$ 'usually' didn't work when n was not prime. This prompted us to make the bold conjecture:

Conjecture: If n is prime, then $10^{n-1} \underset{n}{\equiv} 1$.

Which we immediately realized was false!, because it fails to be true for 2 and 5. But these primes are 'special'; they divide 10. So we formulated the modified conjecture:

Conjecture: If n is prime and does not divide 10, then $10^{n-1} \underset{n}{\equiv} 1$.

Which, we will see eventually, is true! But at this point our speculations have outrun our toolkit; we need to back up a little and develop some techniques which will give us the ability to verify a statement like this.

To get started, look at the perfect squares:

$0, 1, 4, 9, 16, 25, 36, 49, 64, 81, 100, 121, 144, \ldots,$

and now look at the differences of consecutive squares:

$1, 3, 5, 7, 9, 11, 13, 15, 17, 19, 21, 23, \ldots.$

Such a pattern can hardly be a coincidence, can it? If we think about how to use that sequence (sorry, it's the right term to use...) of differences to 'build' the squares from 0, we have

$1 = (0+)1$
$4 = (0+)1 + 3$
$9 = (0+)1 + 3 + 5$
$16 = (0+)1 + 3 + 5 + 7$

and the pattern continues all the way up through our list, to

$144 = 1 + 3 + 5 + 7 + 9 + 11 + 13 + 15 + 17 + 10 + 21 + 23$.

The question is, does it continue? Our conclusion was that it should, and so we asserted that 'The sum of consecutive odd numbers, starting from 1, is always a perfect square.'

Delving even deeper, comparing the 12 of 12^2 to the $23 = 11 + 12$ at the end, and others, we concluded that the sum of the odd numbers up to $2n - 1$ should be n^2. The question is, how to prove it? For this, we need a new technique, known as the *Principle of Mathematical Induction* (or PMI [not PIM!]) for short). It asserts that if $Q(n)$ is a statement which involves the number n and for some integer n_0 [read by most people as "n-naught", although "n-zero" and "n-sub-zero" are also fairly common] we have

$Q(n_0)$ is a true statement, and

for any $n \geq n_0$, if we know that $Q(n)$ is true then we can prove that $Q(n+1)$ is true, then

$Q(n)$ is a true statement for every integer $n \geq n_0$.

The idea behind this is that knowing $Q(n_0)$ is true (by the 'base case' hypothesis) implies (by the 'inductive case' hypothesis) that $Q(n_0 + 1)$ is true, which in turn implies that $Q(n_0 + 2)$ is true, so $Q(n_0 + 3)$ is true, and so on; repeating this $n - n_0$ times will allow us to reach $Q(n)$, which will then be true! A different perspective, which really pinpoints the 'assumption' we are making, is that if $Q(s)$ is false for some $s \geq n_0$, then there is a smallest such s (which we will call n) for which the statement is false. [This is known as the *Archimedean Principle*: a collection of integers larger than n_0, if it contains any integer at all, has a smallest element.] But now either $n = n_0$ (which can't happen: $Q(n_0)$ is true!), or $n > n_0$, in which case $Q(n - 1)$ is true (since $n - 1 \geq n_0$). But then the inductive case tells you that $Q(n) = Q([n - 1] + 1)$ must be true! Oops... So there can be no smallest n with $Q(n)$ false, so $Q(n)$ can never be false!

This technique allows us to prove our assertion about squares, since $1 = 1^2$ is true (the base case!) and if we suppose that the sum of odd numbers up to $2n - 1$ equals n^2, then to get the sum of odds up to the next odd number, $2n + 1$, we can start with the sum we know, adding up to n^2, and add $2n + 1$, giving $n^2 + 2n + 1 = (n + 1)^2$! [We actually got here by a slightly more roundabout way, recognizing 'FOIL' being written out in front of our eyes...]. More symbolically:

If $1+3+\cdots+(2n-1) = n^2$ [this is our statement $Q(n)$], then $1+3+\cdots+(2(n+1)-1) = 1+3+\cdots+(2n+1) = 1+3+\cdots+(2n-1)+(2n+1) = [1+3+\cdots+(2n-1)]+(2n+1) = n^2 + (2n + 1) = (n + 1)^2$, so $1 + 3 + \cdots + (2(n + 1) - 1) = (n + 1)^2$ [this is our statement $Q(n+1)$]. So since $Q(1)$ is true and $Q(n)$ true implies that $Q(n+1)$ is true, we know that $Q(n)$ is true for all $n \geq 1$, by PMI.

In much the same vein, if we explore the sums of successive cubes, we find that

$1^3 = 1$
$1^3 + 2^3 = 1 + 8 = 9 = 3^2$
$1^3 + 2^3 + 3^3 = 9 + 27 = 36 = 6^2$
$1^3 + 2^3 + 3^3 + 4^3 = 36 + 64 = 100 = 10^2$
$1^3 + 2^3 + 3^3 + 4^3 + 5^3 = 100 + 125 = 225 = 15^2$

and more than that, we discovered, since $3 = 1 + 2$, $6 = 1 + 2 + 3$, $10 = 1 + 2 + 3 + 4$, and $15 = 1 + 2 + 3 + 4 + 5$, we were led to suspect that:

For every integer $n \geq 1$, $1^3 + 3^3 + \cdots + n^3 = (1 + 2 + \cdots + n)^2$

Which we proceeded to try to show by induction! The base case, $1^3 = (1)^2$, is true. So we moved on to look at the inductive case. Suppose that $1^3+3^3+\cdots+n^3 = (1+2+\cdots+n)^2$. Then adding one more cube, we get

$1^3 + 3^3 + \cdots + n^3 + (n+1)^3 = (1+2+\cdots+n)^2 + (n+1)^3$, by the inductive hypothesis. If we write $\Sigma = 1 + \cdots + n$ to save ourselves some writing, what we want to show is that $\Sigma^2 + (n+1)^3 = (1+\cdots+n+(n+1))^2 = (\Sigma+(n+1))^2$. But now FOIL came to the rescue again!

$(\Sigma+(n+1))^2 = \Sigma^2+2(n+1)\Sigma+(n+1)^2$, and for this to be the same as $\Sigma^2+(n+1)^3$, what we need is $2(n+1)\Sigma + (n+1)^2 = (n+1)^3$, or, killing off a factor of $(n+1)$ everywhere, $2\Sigma+(n+1) = (n+1)^2$, which means $2\Sigma = (n+1)^2-(n+1) = n^2+n$. So for our inductive step to succeed, we need to know that

$\Sigma = (1+2+\cdots+n) = \frac{1}{2}(n^2+n)$ (for every integer $n \geq 1$).

Which we can show is true! By induction! We will leave this verification for your thought problem for next time...