Math 203
Topics for Chapter 2: Tilings

To do:

Understand the relationship among the interior angles of a polygon; decribe how this relates to tilability.

Understand how to tile the plane with regular polygons and irregeular triangles and quadrilaterals.

Find the symmetries in a figure and in a tiling. Construct tilings of strips with particular symmetries, find the crystallographic symbol of a tiling.

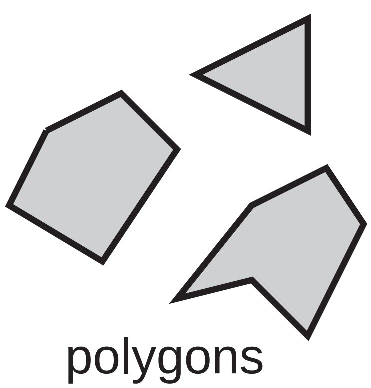
Understand how to alter a tile while maintaining its tilability (a la Escher).

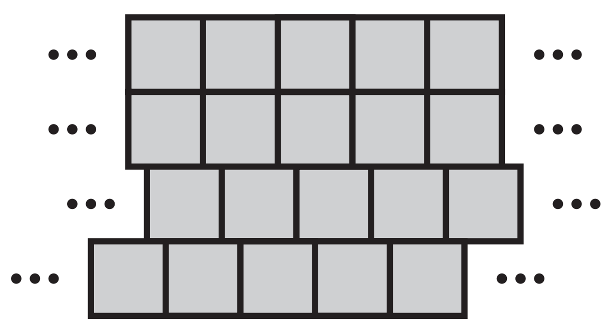
A *tiling* is a way of covering up something (usually the whole plane) with a limited number of shapes. The shapes (which we call *tiles*) aren't allowed to overlap, and every point in the plane must get covered by one of the tiles.

The main question is: what collections of shapes can actually tile the plane? Some practical applications of this include designing machinery for stamping out shapes from sheets of metal; the most efficient shape to use would be one that tiles, since the shapes could be cut out of the sheet with no wasted space. Traditionally, though, tilings have been studied as much for aesthetic reasons as any other, as a way to create interesting patterns.

We will start small, by first considering shapes that are *polygons*; the boundary of the shape is a (finite) number of *edges* joined end-to-end, meeting at their *vertices*. A polygon is described by the number of its edges; with 3 it is a *triangle*, with 4 a *quadrilateral*, then *pentagon, hexagon, septagon, octagon, nonagaon*, and *decagon*. If a polygon has n sides, we can also call it an *n-gon* (as in '18-gon').

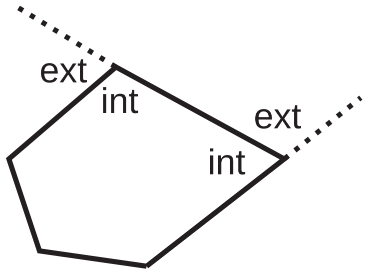
Then we will start even smaller, by considering tilings the use only one shape; these are called *monohedral tilings*. For example, with a square, we can stack them one on top of another, to tile an (infinite) strips, then stack strips side by side to tile the whole plane. But in fact, in some sense we can build 'too many' tilings this way, since the strips can slide along one another to create infinitely many different-looking tilings.

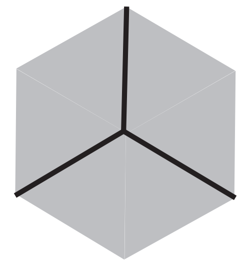




So we make even more restrictions; we try only to build *edge-to-edge* tilings, where the edge of one tile is always glued to the entire edge of the adjacent tile.

At each vertex of a polygon there are two angles worth keeping track of, the *interior angle* and the *exterior angle*. Angles are measured in degrees; a full circle is 360 degrees. So 60 degrees, for example means 60/360, or one-sixth, of a full circle. The interior angle is the angle you see inide of the tile; the exterior angle, is basically, the angle you add to the interior one to get 180 degrees. A polygon is *convex* if the interior angles are all less than 180; essentially, while travelling along the edges, we always turn the same direction at the vertices. The sum of the interior angles of a triangle is 180 degrees, and since a (convex) m-gon can be cut into $(m-2)$ triangles, the sum of the interior angles of a (convex) m-gon is always $(m-2)\times 180$. (This result also holds true for non-convex polygons.)





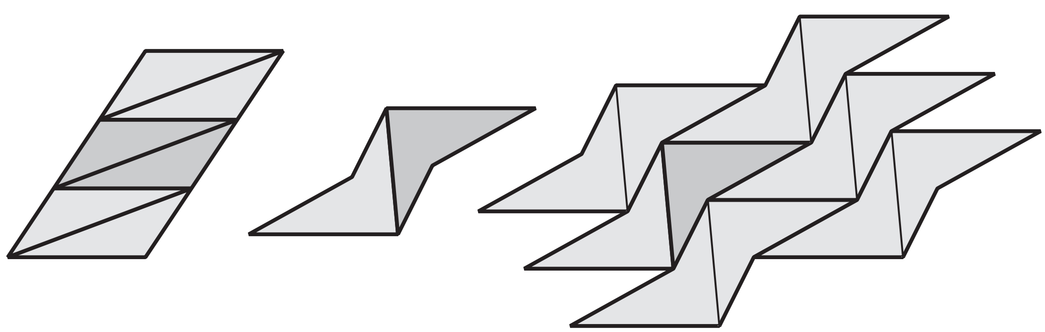
If we restrict our attention still further, to *regular* polygons, ones that have all edges the same length and all interior angles the same, then we can answer the question of which regular polygons have a monohedral, edge-to-edge tiling of the plane, using just information about the interior angles. This is because of the fact (that we will use again and again) that in order for polygons to tile the plane, at every vertex the tiles have to fit together to go around *exactly once*. So the sum of the interior angles meeting at a vertex has to be 360 degrees. But for a regular polygon, all of these angles are the same, so the interior angle has to *evenly divide* 360. Since this angle is $1/m$-th of the sum for the regular m-gon, or $(m-2)\times 180/m$, all we have to do is figure out which of these evenly divide 360 ! If we check, we find that we get (here we'll write (number of sides):(interior angle)

3:60 , 4:90 , 5:108 , 6:120 , 7:128+4/7 , 8:135 , 9:140, 10:144, 11:147+3/11 , 12:150 , 15:156 , 18:160 , 24:165 , etc.

In fact, for anything bigger than 6, the interior angle will be between 120 (which goes into 360 3 times) and 180 (which goes into 360 two times), and so could **never** evenly divide 360. So looking at our list, only 3,4, and 6 **might** work. In fact they all do, so only the equilateral triangle, square, and regular hexagon admit regular, monohedral, edge-to-edge tilings of the plane.

If we now relax things a little, to consider *irregular* polygons, the question is: what irregular polygons tile the plane? Here the basic idea is that we can see that polygons with some number of sides tile because other *special* polygons with more sides tile...

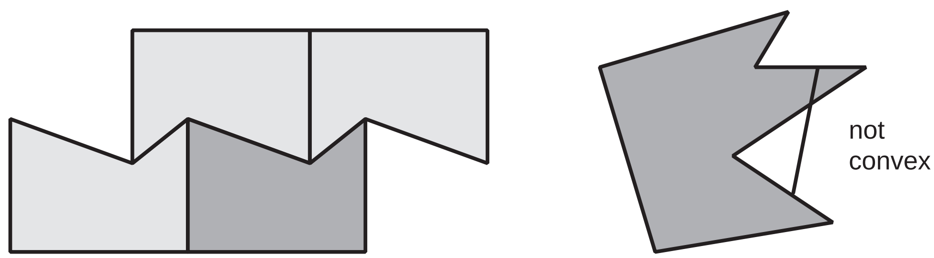
Start with a triangle. Does every triangle tile the plane? The answer is 'Yes', because if we take two of them and glue them together by rotating one of them around a side and glung the two sides together, we get a *parallelogram*. We can stack parallelograms together side by side (just like with a square) to tile an (infinite) strip, and then stack strips to ile the whole plane. So this is 'really' a tiling by parallelograms (a special kind of quadrilateral), with each paralellogram made up of two of our original triangles.



Similarly, it turns out that any quadrilateral can tile the plane. This is because if we take two of them, rotate one around an edge and glue the two together, they make a special kind of hexagon called a *par-hexagon*; it has three pairs of parallel edges. These can be stacked (just like a regular hexagon) on top of one another to tile a strip, and these strips can be stacked side-by-side to tile the whole plane. So the tiling by our quadrilaterals is 'really' a tiling by par-hexagons, each of which is made up of two of our quadrilaterals.

After that, things get a little more complicated. Not all pentagons tile (the regular one doesn't!), although some do. For example, take a rectangle and grab a point in the middle of one side and pull; this will tile (gluing two together like above you get an octagon that happens to tile a strip...).

Some hexagons tile, like the par-hexagons; in fact, there is a complete discription of which hexagons actually tile the plane. Suprisingly, there is **no** such list for the pentagons! Nobody knows (yet) exactly which pentagons tile.

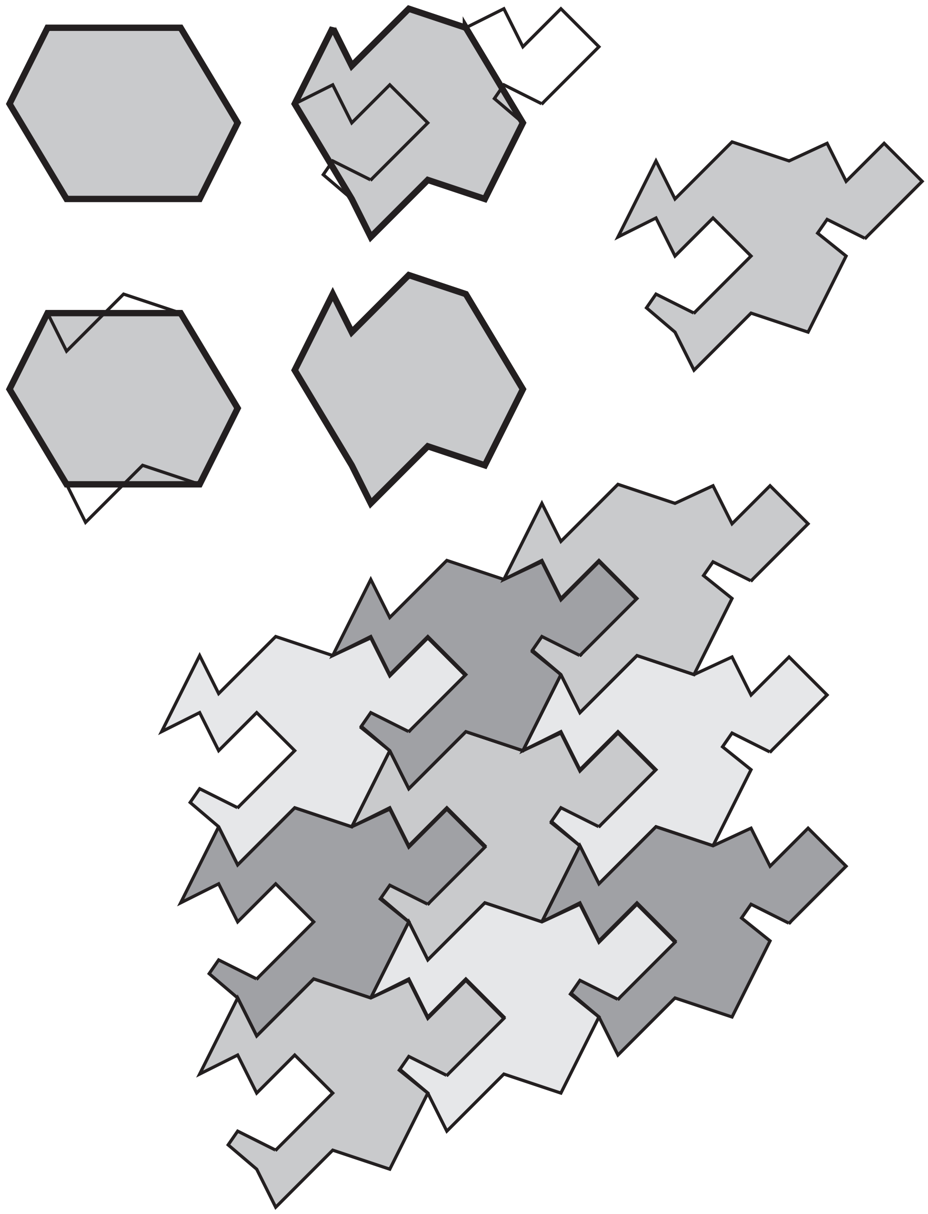


For seven or more sides, it turns out that no *convex* m-gon, with m bigger than 6, can tile the plane. Non-convex polygons with 100,000 sides which tile the plane are not too difficulat to build; we'll see how shortly!

We can build (monohedral) tilings with more complicated tiles, by using simpler patterns. We'll do this by insisting that we don't turn any tiles around, as we did with triangle, quadrilaterals, and even the pentagon above. Every polygon in the tiling will be a *translation* of every other tile, i.e., we can move one to the other without nghe tile at all. So, for example, our tilings by par-hexagons are OK, each tile is a translate of the others. The tiling by parallelograms underlying our tiling by triangles is also a *tiling by translation.* The whole idea is that if we look at the two sides of our polygon that correspond to sides getting glued together in the tilng, they must be *parallel* to one another in the polygon, because they end up being glued together without any turning of the tiles. This means that the sides of our polygon have to be paired up; there have to be lots of parallel sides.

The basic idea behind building lots of tilings by translation is to start with something that works, and then alter the tile by changing both pair of sides that are parallel, to get something else that tiles by translation. The idea is that if we break one of the two sides into two sides, by introducing a 'kink', we have to do the **same thing** to the other side, to make sure that they will still match up. In this way, we can go from something we *know* tiles by translations, to something much more complicated, which also tiles by translations! This is the basis for many of the prints by M.C. Escher, which you can find in the textbook and elsewhere.

The essential idea is to replace a side by some more complicated curve (it need not be straight lines), and then replace the side opposite on your starting polygon by the same curve. In fact, since you have to start with something that tiles by translations, you can always (I think....) make your starting point either a parallelogram or a par-hexagon. The figure below shows an example of this.



In this way we can build from a tilable tile that is fairly simple to something much more complicated, which still tiles the plane.

We can help to understand a tiling by understanding its *symmetries*: rigid motions of the plane which take the tile (or tiling) back to itself. That is, the motion moves individual tiles, but after the motion the tiling looks exactly the same. There are four basic kinds of symmetries we can encounter:

Translation: sliding everything in the same direction. A translation is completely described by giving the direction of motion, and the distance that points will be slid in that direction.

Rotation: individual points move around in a circle around a center point. A rotation is completely decribed by giving the center point (which does not move) and the angle that a line out from the center moves through.

Reflection: points are flipped across a line, which is unmoved by the relection. A reflection is completely described by the line that we reflect across.

Glide reflection: This is essentially reflection across a line followed by a translation in the direction of the line we reflected across. Individual point flip across the line and then slide in a direction parallel to the line. a glide reflection is completely described by the line of reflection and the distance we slide in the line's direction.

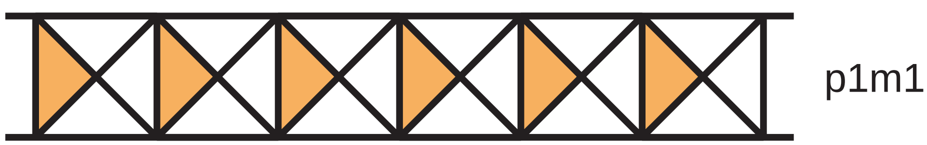
All of our regular tilings have many different kinds of symmetries, e.g., rotations around vertices of differing degrees (multiples of 60 degree for triangles, of 90 for squares, and of 120 for hexagons). For this discussion we will focus on tilings of infinite horizontal strips, and explore what combinations of symmetry types can occur for a single tiling. We will allow ourselves to color tiles different colors (to represent tiles that aren't to be taken one to the other under a symmetry), so in essence we are really studying the symmetries of the patterns we are imposing. We will use the tiling of a strip by triangles (forming larger squares within the tiling) below for our explorations, but many other tilings and patterns are equally instructive.

We will codify the kinds of symmetries in a pattern using a 4-character symbol ABCD first introduced in the study of crystals. This crystallographic symbol represents the four kinds of symmetries we might find. The first, A, encodes the presence of translational symmetry, and takes the value "p" if there is a translation symmetry, and "1" if there isn't. The second, B, is for reflection symmetry across vertical lines, and has the value "m" if there is a vertical reflection symmetry, and "1" if there isn't. The third, C, encodes horizontal reflection and glide reflection symmetry; it takes the value "m" if there is a reflection symmetry across a horizontal line, "a" if there is no reflection symmetry but there is a glide reflection symmetry, and "1" if there is neither. Finally, the fourth, D, encodes rotational symmetry. (For a strip this is necessarily a 180 degree rotation; nothing else can take the horizontal strip back to itself.) It takes the value "2" if there is a rotational symmetry, and "1" if there isn't.

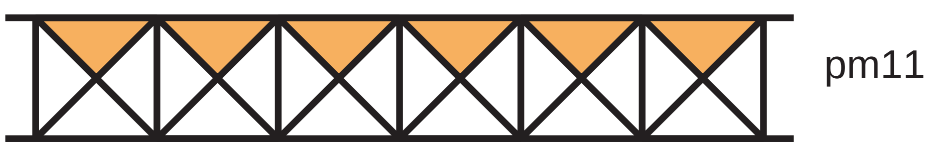
It turns out that of all the combinations of symbols we could make, there are exactly seven which occur as the symmetries of a strip pattern with translational symmetry (i.e., which start with "p"). They are the symbols

pmm2 , pma2 , pm11 , p1m1 , p1a1 , p112 , and p111 .

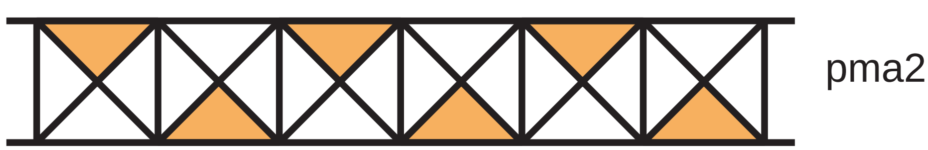
There are several ways in which we can "take away" a particular kind of symmetry by coloring our pattern in certain ways. For example, to take away a vertical reflection symmetry, we can introduce colored tiles the point in one direction and not the other.



We can take away horizontal reflection symmetry by coloring tiles on top differently than on the bottom.



We can take away horizontal reflection, but keep glide reflection, by coloring tiles on top and bottom opposite to one another, but let them alternate.



Similarly, one can build patterns which realize each of the remaining symbols; we leave that to you to work out!