Math 203

Topics for Chapter 11: Social Choice

The basic question: what is the best way to choose the best of several alternatives, based on the preferences of some group of people? For example, how do a group of people choose which restaurant to eat dinner at, based on the culinary preferences of the people in the group? We explore several different such 'voting systems', exploring their advantages and disadvantages.

In most cases, the voting system uses a *preference list* from each voter: a list of the alternatives in the order that the voter would prefer them, from highest preference to lowest. In most case, we also allow *ties*; more than one alternative can be declared the 'winner'. Just about the only condition we impose on a voting system is that no matter what collection of preference lists we give it, it will always declare at least one alternative to be a winner (we have to eat *somewhere*; someone needs to be President (right?)).

For a vote between only two alternatives, we can use **majority rule**: the alternative with the most first place votes is the winner. In this case, we often think of a preference list as one name (our top choice); the name not appearing would be our second (i.e., last) choice.

Majority rule has (at least) three properties that are considered desirable:

(1) All voters are treated equally. It doesn't matter who turns in a ballot, it will be treated the same as anybody else's ballot.

[[Compare this with a **dictatorship**: whatever the dictator votes for wins. This fails to have property (1).]]

(2) Both candidates are treated the same. It doesn't matter who the candidates are, if everybody switched their preference order, the winner would lose and the loser would win.

[[Compare this with **imposed rule**: no matter how people vote, one pre-chosen alternative will win. This fails to have property (2).]]

(3) If someone changes their votes from the loser to the winner, the winner will still win.

It is a fact (called May's Theorem) that the only voting systems for two alternatives that satisfy all three of these conditions are majority rule and the system that says that both alternatives always win!

When we move on to consider voting systems for three or more alternatives, we find that there are several that are used in everyday life.

A **plurality vote** looks only at each person's top choice; the choices with the most votes are the winners. (Note that, in practice, ties are actually rare; among one million people, it is extremely unlikely that *exactly* 300,000 (say) will vote for two candidates.) With three or more alternatives, it is therefore possible to win without receiving a majority of the votes.

(Sidebar: Majority rule is actually sometimes used, as well; for example, when each party chooses its presidential candidate at the national convention. These days, someone usually wins on the first ballot, but in past campaigns, several ballots were required, a few people changing their votes each time, before someone managed to coerce enough people to vote for him/her to get a majority! Choosing a new pope is similar, but at least two-thirds of the votes are required.)

But the plurality vote system has a drawback, which is illustrated by the following collection of preference list. We have 3 alternatives A,B,C , and the following numbers of voters rank them in the following orderings:

22 for A–B–C	23 for A–C–B
15 for B–A–C	29 for B–C–A
7 for C–B–A	4 for C–A–B

With a plurality vote, A gets 45 votes, B gets 11 and C gets 44, so A wins. But! it turns out that if either A or B dropped out of the race, then based on the preference lists, in an A/B race, B gets 51 votes to A's 49, and in a B/C race, B gets 66 votes to C's 34. So B is preferred to both A and C, considered one at a time!

We would call B the **Condorcet winner** of the election, based on our voting scheme. This means that if we used our voting system on our preference lists, but erase every alternative but B and one other alternative, then B will always win over each other alternative. A voting systen satisfies the **Condorcet Winner Criterion** (CWC, for short) if for any collection of preference lists we might come up with, **if** there is a Condorcet winner based on our voting system, then this alternative is also a winner based on the voting system alone. Since in the above example, the winner under the plurality vote (A) was not the same as the Condorcet winner (B), plurality voting does not satisfy CWC.

It is interesting to note that both dictatorship and imposed rule *do* satisfy CWC! We should also point out that in most cases, a collection of preference lists will **have no Condorcet winner**. But when it does, CWC says that alternative should be the same as the winner from the voting system.

In the next voting system, the goal is not only to find a winner, but to rank all of the alternatives. The basic idea is to 'average' everyone's preference lists, to get one big preference list. This method is called the **Borda count**. The idea is, for each preference list, to assign a *weight* to each alternative in the list, based on how high it is on the list. In a traditional Borda count, if there are n choices, the top choice is given a weight of $n-1$, the second choice gets $n-2$, all the way down the list, so the last choice gets weight 0. A total weight is then derived by adding, for each alternative, the weights they have received from each list.

For example, if we have three alternatives A,B, and C, and the following numbers of people rank them the following ways:

6 have A–B–C	5 have B–A–C
4 have A–C–B	4 have B–C–A
2 have C–A–B	2 have C–B–A

then the final vote will give a ranking of A–B–C, since A receives $6\times2+5\times1+4\times2+4\times0+2\times1+2\times0 = 27$ total points, A receives $6\times1+5\times2+4\times0+4\times2+2\times0+2\times1 = 26$ points, and A receives $6\times0+5\times0+4\times1+4\times1+2\times2+2\times2 = 15$ points.

But strange things happen if we look at the last two voter's preference lists. Their last choice, A, won. If they really hated A, they could at the very least let B win instead of A, by changing their votes to B–C–A. Then A gets 27, B gets 28 and C gets 13 ! In fact, if only one of them made this change, A and B would tie for first place. The strange part of this is that these two turned A into a loser and B into a winner, without changing the order in which they preferred A and B!

A voting system is said to satisfy **Independence of Irrelevant Alternatives (IIA)** if no single voter, by changing his/her vote, can turn a losing alternative (B) into a winning alternative, without reversing the order in which they rank B with one of the previously winning alternatives. Put differently, moving only an irrelevant alternative (one that isn't a winner) around in the list cannot make a losing alternative win.

The example above shows that the Borda count does not satisfy IIA. The same lists also show that the plurality vote fails to satisfy IIA; the same change turns winner A and loser B into winner B and loser A! Interestingly, Both dictatorship and imposed rule satisfy IIA! If a system fails IIA, the outcome of the vote can be manipulated by unscrupulous sorts by voting 'strategically', basically, in a way that insures your higher choice wins, even if your ('real') top choice won't.

Many voting siystems are similar to the Borda count, but just weight the list differently. For example, a common weighting system is 10,5,3,2,1,0,0,0 (dropping by about half each time). Technically, these aren't called Borda counts, although they are 'Borda-like'.

Our next voting system is frequently used in legislatures, and is called **Sequential Pairwise Voting**. The idea is that a predetermined 'agenda' (a list of the alternatives) is used to allow each voter to vote on only two candidates at a time, with the one receiving the majority being the winner. We start with the top two from the agenda, and the winner between those two goes on to face the next alternative on the agenda. The winner between those two goes on to face the next, and this continues until we reach the bottom of the agenda. We can think of this as a game show where each day a champion faces a challenger; whoever wins becomes the next day's champion, to face a new challenger. When we run out of challenger's, the remaining chamption is the winner.

A basic flaw with this system is that the last item on the agenda has to beat only one other alternative in order to be the winner. For example with four alternatives and preference lists

3 for A–D–C–B   2 for B–A–D–C

2 for C–B–A–D

and the agenda A–B–C–D, then in A vs. B, B wins 4 to 3 (i.e, 4 prefer B over A, and 3 preferA over B), so faces C, the next alternative in the agenda, where C wins 5 to 2, so C faces D, where D wins 5 to 2. So D is the winner of the vote, based on sequential pairwise voting.

But! we could have made **any** alternative win, by choosing a different agenda. For A to win, use the agenda B–C–D–A, for B use C–D–A–B, and for C use D–A–B–C ! In this case what this means is that the person who sets the agenda can rig things so that whoever they want to win will win!

In our beginning example, there's something else wrong as well. Alternative D won, but a look at every preference list shows that *every voter preferred* A *over* D ! This means that sequential pairwise voting fails to satisfy something called the **Pareto condition**. A voting system satisfies the **Pareto condition** if an alternative A cannot be one of the winners when *every voter* prefers the *same* other alternative, B, to A.

It turns out that dictatorship, imposed rule, plurality vote, and Borda counts *do* satisfy the Pareto condition. Which has to make you wonder why sequential pairwise voting is used so often in state and national legislatures?

The final system we will look at is called the **Hare system**. In this system, we repeatedly look at the top candidate on each preference list. The one (or one's) receiving the fewest votes is removed from everybody's list. We continue to do this until either only one alternative remains (who is then the winner), or every remaining choice is tied for last place among the top choices (in which case we declare them all to be winners). For example, with the preference lists

5 for A–B–C	4 for B–C–A
3 for C–B–A	1 for C–A–B

in the first round A gets 5 votes, B gets 4 and C get 4, so both B and C are eliminated. This leaves A as the only alternative, so A is the winner.

But in a slightly different vote,

5 for A–B–C	4 for B–C–A
3 for C–B–A	1 for A–C–B

first A gets 6, B gets 4 and C gets 3, so C, having the fewest votes, is eliminated. Removing C from every list, we get the new lists

5 for A–B	4 for B–A
3 for B–A	1 for A–B

In the second round, then A gets 6 votes, and B gets 7, so A is eliminated, leaving B the only alternative, so B wins.

But wait! There is something terribly wrong here. The only change in these two collections of lists is that in the second case, the last voter moved A *up* in his/her list. This change, however, resulted in A **losing**! As a result, we say that the Hare system fails to satisfy **monotonicity**: a voting system satisfies **monotonicity** if whenever an alternative A is among the winners, and a voter then moves A *higher* in their preference list, then the voting system will still declare A to be a winner.

Plurality vote, Borda counts, sequential pairwise voting, dictatorship, and imposed rule all satisfy monotonicity. The fact that having someone rank an alternative higher can cause that alternative to lose, in the Hare system, is considered a serious enough defect that nobody really uses the Hare system (except for hiring decisions in the math department at NMSU...). The only reason someone might prefer the Hare system is that it does insure that the eventual winner was the top choice of *some* voter (since without a top vote, you are eliminated in the first round (zero is the lowest vote count you could get...)).

It is not hard to construct collections of preference lists where these different voting systems give different winners. For example, in the second collection of lists in the discussion of the Hare system, A would win the plurality vote. It would also win with the first set, although B would win with a Borda count.

It turns out that there is **no** voting system which can be considered fair, in the sense that it satisfies all of the conditions we have considered (including (1), (2), and (3) from majority rule). In fact, if we simply insist on having a voting system that satisfies both CWC and IIA, then (unless we allow *everyone* to win!) the only voting systems we could use would be dictatorship or imposed rule! Both of which have their own little problems.....

One final voting system is becoming more and more popular; it is called **approval voting**. The idea is that each voter casts a vote for *each* alternative that they find acceptable, i.e, approve of. The alternative that receives the most votes (i.e., is acceptable to the most voters) is the winner. This is in fact a 'Borda'like' count; the weights used are $1, \ldots, 1, 0, \ldots, 0$, where the 1's you approve of and the 0's you don't. Asking whether or not this system satisfies IIA is a little difficult, because the repeated numbers mean that most alternatives are 'tied', so reversing the order of alternatives doesn't really make sense. But it does satisfy CWC, the Pareto condition, and monotonicity. It also satisfies (1), (2), and (3)!