Solutions

Name:

Math 208H, Section 2

Exam 1

1. (15 pts.) Find the length of the curve C given by the parametrization

$$\gamma(t) = (t^2 \cos t, t^2 \sin t) \qquad 0 \le t \le 2\pi$$

$$= (x(t), y(t))$$

$x'(t) = 2t\cos t - t^2 \sin t$

$y'(t) = 2t\sin t + t^2 \cos t$

$(x'(t))^2 + (y'(t))^2 = (2t\cos t - t^2 \sin t)^2 + (2t\sin t + t^2\cos t)^2$

$= 4t^2\cos^2 t - \cancel{4t^3 \sin t\cos t} + t^4 \sin^2 t + 4t^2 \sin^2 t + \cancel{4t^3 \sin t \cos t} + t^4 \cos^2 t$

$= 4t^2(\cos^2 t + \sin^2 t) + t^4(\sin^2 t + \cos^2 t)$

$= 4t^2 + t^4$

$u = t^2 + 4$

$du = 2t\,dt$

$t\,dt = \frac{1}{2}du$

$t=0 \quad u = 4$

$t = 2\pi \quad u = 4\pi^2+4$

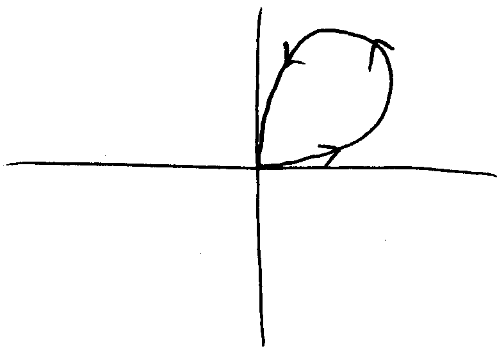
$$\text{Arclength} = \int_0^{2\pi} (t^4 + 4t^2)^{1/2}\,dt = \int_0^{2\pi} t\,(t^2+4)^{1/2}\,dt$$

$$= \frac{1}{2}\int_4^{4\pi^2+4} u^{1/2}\,du = \frac{1}{2}\cdot\frac{2}{3}u^{3/2}\Big|_4^{4\pi^2+4}$$

$$\boxed{= \frac{1}{3}\left((4\pi^2+4)^{3/2} - 4^{3/2}\right)} = \frac{8}{3}\left((\pi^2+1)^{3/2} - 1\right)$$

2. (15 pts.) Find the area of the region lying inside of a single "petal" of the 4-petaled rose

$$r = \sin(2\theta) \text{ , } 0 \leq \theta \leq \frac{\pi}{2}$$



$$\text{Area} = \int_0^{\frac{\pi}{2}} \frac{1}{2}\left(\sin(2\theta)\right)^2 d\theta$$

$$= \int_0^{\frac{\pi}{2}} \frac{1}{2}\left(\frac{1}{2}\left(1 - \cos(2(2\theta))\right)\right) d\theta$$

$$= \int_0^{\frac{\pi}{2}} \frac{1}{4} - \frac{1}{4}\cos(4\theta)\, d\theta = \frac{1}{4}\theta - \frac{1}{16}\sin(4\theta)\Big|_0^{\frac{\pi}{2}}$$

$$= \left(\frac{\pi}{8} - \frac{1}{16}\sin(2\pi)\right) - \left(0 - \frac{1}{16}\sin 0\right)$$

$$= \left(\frac{\pi}{8} - 0\right) - (0 - 0) = \boxed{\frac{\pi}{8}}$$

3. (20 pts.) Find the (rectangular) equation of the line tangent to the graph of the polar curve

$$r = 3\sin\theta - \cos(3\theta)$$

at the point where $\theta = \dfrac{\pi}{4}$.

$$x = r\cos\theta$$
$$= 3\sin\theta\cos\theta - \cos\theta\cos 3\theta$$

At $\theta = \frac{\pi}{4}$,

$$y = r\sin\theta$$
$$= 3\sin^2\theta - \sin\theta\cos 3\theta$$

$$x = 3\sin(\tfrac{\pi}{4})\cos(\tfrac{\pi}{4}) - \cos(\tfrac{\pi}{4})\cos(\tfrac{3\pi}{4})$$
$$= 3\,\frac{\sqrt{2}}{2}\cdot\frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{2}\left(-\frac{\sqrt{2}}{2}\right)$$
$$= \frac{3}{2} + \frac{1}{2} = 2$$

$$y = 3\left(\sin(\tfrac{\pi}{4})\right)^2 - \sin\tfrac{\pi}{4}\cos(\tfrac{3\pi}{4}) = 3\left(\tfrac{\sqrt{2}}{2}\right)^2 - \tfrac{\sqrt{2}}{2}\left(-\tfrac{\sqrt{2}}{2}\right) = \tfrac{3}{2} + \tfrac{1}{2} = 2 .$$

$$\frac{dx}{d\theta} = 3\cos\theta\cos\theta + 3\sin\theta(-\sin\theta) - \left(\sin\theta\cos 3\theta + \cos\theta(-3\sin 3\theta)\right)$$
$$= 3\cos^2\theta - 3\sin^2\theta + \sin\theta\cos 3\theta + 3\cos\theta\sin 3\theta$$
$$= 3\left(\tfrac{\sqrt{2}}{2}\right)^2 - 3\left(\tfrac{\sqrt{2}}{2}\right)^2 + \left(\tfrac{\sqrt{2}}{2}\right)\left(-\tfrac{\sqrt{2}}{2}\right) + 3\left(\tfrac{\sqrt{2}}{2}\right)\left(\tfrac{\sqrt{2}}{2}\right) = \tfrac{3}{2} - \tfrac{3}{2} - \tfrac{1}{2} + \tfrac{3}{2}$$
$$= 1$$

$$\frac{dy}{d\theta} = 3\cdot 2\sin\theta\cos\theta - \left(\cos\theta\cos 3\theta + \sin\theta(-3\sin 3\theta)\right)$$
$$= 6\sin\theta\cos\theta - \cos\theta\cos 3\theta + 3\sin\theta\sin 3\theta$$
$$= 6\left(\tfrac{\sqrt{2}}{2}\right)\left(\tfrac{\sqrt{2}}{2}\right) - \left(\tfrac{\sqrt{2}}{2}\right)\left(-\tfrac{\sqrt{2}}{2}\right) + 3\left(\tfrac{\sqrt{2}}{2}\right)\left(\tfrac{\sqrt{2}}{2}\right) = \tfrac{6}{2} + \tfrac{1}{2} + \tfrac{3}{2} = 5$$

$$\frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta} = \frac{5}{1} = 5$$

Tangent line: $\boxed{y - 2 = 5(x-2)}$

4. (15 pts.) Find perpendicular vectors $\vec{u}$ and $\vec{v}$, one of which points in the same direction as $\vec{w} = (2,3,5)$, whose *difference* is $\vec{z} = (2,2,1)$.

want $\vec{u} \perp \vec{v}$, $\quad \vec{u} - \vec{v} = (2,2,1) \qquad \vec{w} = (2,3,5)$

$\vec{u}$ or $\vec{v} = c\vec{w}$ $(c>0)$. So want $c = \dfrac{(2,2,1)\cdot(2,3,5)}{\|(2,3,5)\|^2}$

$$= \frac{4+6+5}{(4+9+25)} = \frac{15}{38}$$

$$\vec{u} = c\vec{w} = \frac{15}{38}(2,3,5) = \left(\frac{30}{38}, \frac{45}{38}, \frac{75}{38}\right)$$

$$\vec{v} = \vec{u} - (2,2,1) = \left(\frac{30}{38} - 2, \frac{45}{38} - 2, \frac{75}{38} - 1\right)$$

$$= \left(\frac{30-76}{38}, \frac{45-76}{38}, \frac{75-38}{38}\right)$$

$$= \left(\frac{-46}{38}, \frac{-31}{38}, \frac{37}{38}\right)$$

Check! $u \perp v$? $\quad (30)(-46) + (45)(-31) + (75)(37)$

$$= -1380 - 1395 + 2775 = -2775 + 2775 = 0 \checkmark$$

$$\begin{array}{r} 45 \\ \times\, 31 \\ \hline 45 \\ 135 \end{array} \qquad \begin{array}{r} 75 \\ \times\, 37 \\ \hline 525 \\ 225 \\ \hline 2775 \end{array}$$

$$\boxed{\begin{array}{l} u = \frac{15}{38}(2,3,5) = \left(\frac{30}{38}, \frac{45}{38}, \frac{75}{38}\right) \\ v = \left(\frac{-46}{38}, \frac{-31}{38}, \frac{37}{38}\right) \end{array}}$$

$$u - v = \left(\frac{76}{38}, \frac{76}{38}, \frac{38}{38}\right) = (2,2,1) \checkmark$$

5. (15 pts.) Show that if the vectors $\vec{v}=(a_1, a_2, a_3)$ and $\vec{w}=(b_1, b_2, b_3)$ have the same length, then the vectors

$$\vec{v} + \vec{w} \text{ and } \vec{v} - \vec{w}$$

are perpendicular to one another.

$$\|v\| = (a_1^2+a_2^2+a_3^2)^{1/2} = \|w\| = (b_1^2+b_2^2+b_3^2)^{1/2}$$

So $a_1^2+a_2^2+a_3^2 = b_1^2+b_2^2+b_3^2$ (*)

$$v+w = (a_1+b_1, a_2+b_2, a_3+b_3)$$
$$v-w = (a_1-b_1, a_2-b_2, a_3-b_3)$$

$$\begin{aligned}(v+w)\cdot(v-w) &= (a_1+b_1)(a_1-b_1) + (a_2+b_2)(a_2-b_2) + (a_3+b_3)(a_3-b_3) \\ &= a_1^2-b_1^2+a_2^2-b_2^2+a_3^2-b_3^2 \\ &= (a_1^2+a_2^2+a_3^2) - (b_1^2+b_2^2+b_3^2) \\ &= 0 \quad \text{b/c of (*)}\end{aligned}$$

So $\boxed{v+w \perp v-w}$.

OR
$$\begin{aligned}(v+w)\cdot(v-w) &= v\cdot v + w\cdot v - v\cdot w - w\cdot w \\ &= v\cdot v + v\cdot w - v\cdot w - w\cdot w \\ &= v\cdot v - w\cdot w \\ &= \|v\|^2 - \|w\|^2 = 0 \quad \text{b/c } \|v\| = \|w\|.\end{aligned}$$

6. (20 pts.) Find the equation of the plane in 3-space which passes through the three points

$$\overset{P}{(1,2,1)}\ ,\ \overset{Q}{(6,1,2)},\ \text{and}\ \overset{R}{(9,-2,1)}\ .$$

Does the point $(3,2,1)$ lie on this plane?

$\vec{v} = PQ = (6-1, 1-2, 2-1) = (5,-1,1)$

$\vec{w} = PR = (9-1, -2-2, 1-1) = (8,-4,0)$

$N = \vec{v} \times \vec{w} = \big((-1)(0)-(1)(-4), -((5)(0)-(1)(8)), (5)(-4)-(-1)(8)\big)$

$= (0+4, -(0-8), (-20+8)) = (4,8,-12)$

$=$ normal to the plane.

$N \cdot (x,y,z) = N \cdot P$

$(4,8,-12)(x,y,z) = (4,8,-12)\cdot(1,2,1)$

$4x+8y-12z = 4+16-12 = 8$

$\boxed{4x+8y-12z = 8}$ (or $x+2y-3z=2$)

$(3,2,1)$ on plane?

$4(3)+8(2)-12(1) \overset{?}{=} 8$

$12+16-12 = 16 \neq 8$ $\boxed{\text{NO.}}$