Solutions

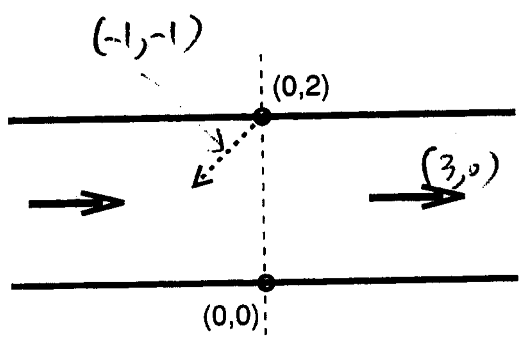
Name:

Math 208H, Section 002

Exam 2

Show all work. How you get your answer is just as important, if not more important, than the answer itself. If you think it, write it!

1. (15 pts.) You are on the north side of a 2 mile wide river, which is flowing from west to east at 3 miles per hour. You cross the river in a boat, which you keep pointed in a southwesterly direction, at a steadily increasing velocity of $\sqrt{2}(1+6t)$ miles per hour. Where will you finally hit the southern bank of the river?



$s(t) = ?$ $\quad s(0) = (0,2)$

$$v(t) = \sqrt{2}(1+6t)\frac{(-1,-1)}{\|(-1,-1)\|} + (3,0)$$

$$= (3-(1+6t),\, 0-(1+6t))$$

$$= (2-6t,\, -1-6t)$$

$\|(-1,-1)\| = \sqrt{1+1} = \sqrt{2}$

$$s(t) - s(0) = \int (2-6t, -1-6t)\,dt = (2t-3t^2, -t-3t^2)$$

$$s(t) = (0,2) + (2t-3t^2, -t-3t^2) = (2t-3t^2,\, 2-t-3t^2)$$

$$= (x(t), y(t))$$

What is $x(t)$ when $y(t)=0$?

$2-t-3t^2=0 = (2-3t)(1+t)$ $\quad t=2/3,\ t=-1$ not realistic

At $t=2/3$ $\quad x(t) = 2(2/3) - 3(2/3)^2 = 4/3 - 4/3 = \underline{0}$

You hit the south bank at $\underline{(0,0)}$.

2. (15 pts.) Find the second partial derivatives of the function

$$f(x,y)=\frac{\cos(xy)}{x}$$

$$f_x = \frac{x(-\sin(xy)(y)) - 1\cdot\cos(xy)}{x^2} = \frac{-xy\sin(xy)-\cos(xy)}{x^2}$$

$$f_y = \frac{1}{x}\left(-\sin(xy)(x)\right) = -\sin(xy)$$

$$f_{xx} = -\frac{\left(2x(-xy\sin(xy)-\cos(xy)) - x^2\left((-y\sin(xy) - xy\cos(xy)\cdot y) - (-\sin(xy)\cdot y)\right)\right)}{x^4}$$

$$f_{xy} = \frac{1}{x^2}\left(\cancel{-x\sin(xy)} - xy(\cos(xy)\cdot x) \cancel{- (-\sin(xy)\cdot x)}\right)$$

$$= -y\cos(xy)$$

$$f_{yx} = -\cos(xy)\cdot y = -y\cos(xy)$$

$$f_{yy} = -\cos(xy)\cdot x = -x\cos(xy)$$

$$= -x^{-4}\left(-2x^2y\sin(xy) - 2x\cos(xy) + x^2y\sin(xy) + x^3y^2\cos(xy) - x^2y\sin(xy)\right)$$

$$= x^{-4}\left(-x^3y^2\cos(xy) + \cancel{3}\,2x^2y\sin(xy) + 2x\cos(xy)\right)$$

3. (15 pts.) Find the equation of the tangent plane to the graph of the function

$$f(x,y) = x\ln(x^2+e^y)$$

at the point $(2,0,f(2,0))$.

$f(2,0) = 2\ln(5)$

$$f_x = \ln(x^2+e^y) + x\left(\frac{2x}{x^2+e^y}\right)$$

$$= \ln(x^2+e^y) + \frac{2x^2}{x^2+e^y}$$

at $(2,0)$: $f_x(2,0) = \ln(5) + \frac{8}{5}$

$$f_y = x\left(\frac{e^y}{x^2+e^y}\right)$$

at $(2,0)$: $f_y(2,0) = 2\left(\frac{1}{5}\right) = \frac{2}{5}$

Tangent line:

$$\boxed{z = 2\ln(5) + \left(\ln(5)+\frac{8}{5}\right)(x-2) + \frac{2}{5}(y-0)}$$

4. ~~3~~ (15 pts.) Let $z = f(x, y)$ be any function with continuous second partial derivatives, and suppose that

$$x = rs \text{ and } y = r + s$$

$$\boxed{\begin{array}{ll} x_r = s & y_r = 1 \\ x_s = r & y_s = 1 \end{array}}$$

Use the chain rule to show that

$$z_{rs} = rz_{xx} + (r+1)z_{xy} + z_{yy} + z_x$$

$$z_{rs} = (z_r)_s$$

$$z_r = z_x x_r + z_y y_r = z_x \cdot s + z_y \cdot 1 = s z_x + z_y$$

$$(z_r)_s = (s \cdot z_x + z_y)_s = (s \cdot z_x)_s + (z_y)_s$$

$$= 1 \cdot z_x + s \cdot (z_x)_s + \left((z_y)_x x_s + (z_y)_y y_s\right)$$

$$= z_x + s\left((z_x)_x x_s + (z_x)_y y_s\right) + \left(z_{yx} \cdot r + z_{yy} \cdot 1\right)$$

$$= z_x + s\left(z_{xx} r + z_{xy} \cdot 1\right) + z_{xy} r + z_{yy}$$

$$= z_x + rs z_{xx} + s z_{xy} + r z_{xy} + z_{yy}$$

$$= rs z_{xx} + (r+s) z_{xy} + z_{yy} + z_x$$

Note: this is also

$$z_{rs} = x z_{xx} + y z_{xy} + z_{yy} + z_x \;\ldots\ldots$$

(b/c $rs = x$, $r+s = y$!)

5. (20 pts.) Find the local extrema of the function

$$f(x,y) = 2x^2 + xy^2 + 4y^2 ,$$

and determine, for each, if it is a local max. local min, or saddle point.

$f_x = 4x + y^2 + 0 = 0$

$f_y = 0 + 2xy + 8y = 0 = 2y(x+4)$

or

$y = 0$ $\quad\quad$ $x = -4$

$4x + 0 = 0$ $\quad\quad$ $-16 + y^2 = 0$

$x = 0$ $\quad\quad$ $y^2 = 16$

$y = \pm 4$

Critical points:

$(0,0)$

$(-4,4)$

$(-4,-4)$

$f_{xx} = 4 > 0$

$f_{xy} = 2y$

$f_{yy} = 2x + 8$

$D = f_{xx}f_{yy} - (f_{xy})^2$

$= 4\cdot(2x+8) - (2y)^2$

At $(0,0)$, $D = 4\cdot 8 - (0)^2 = +32 > 0$ local min ~~saddle~~

$(-4,4)$ $\quad D = 4\cdot 0 - (8)^2 = -64 < 0$, saddle

$(-4,-4)$ $\quad D = 4\cdot 0 - (-8)^2 = -64 < 0$ saddle

6. (20 pts.) Find the point(s) on the ellipse

$$g(x, y) = x^2 + 3y^2 = 4$$

where the function

$$f(x, y) = x - 3y + 4$$

achieves it maximum value.

$1 = 2\lambda \cdot 0 = 0$ NO

$f_x = 1 = \lambda g_x = 2x\lambda$ $\quad \lambda = \frac{1}{2x}$ (or $x=0$?) NO

$f_y = -3 = \lambda g_y = 6y\lambda$ $\quad \lambda = \frac{-1}{2y}$ (or $y=0$?) NO

$x^2+3y^2=4$

$-3 = 6 \cdot 0 \cdot \lambda = 0$ NO

So $\frac{1}{2x} = \frac{-1}{2y}$ $\quad 2y = -2x$ $\quad \boxed{y = -x}$

$x^2 + 3(-x)^2 = 4 = 4x^2$

$x^2 = 1$

$x = 1, -1$

"Interesting" points:

$(1,-1)$, $(-1,1)$

$f(1,-1) = 1+3+4 = 8 \leftarrow$ max

$f(-1,1) = -1-3+4 = 0$

Max value at $(1,-1)$.