Solutions

Name:

Math 208H, Section 002

Exam 3

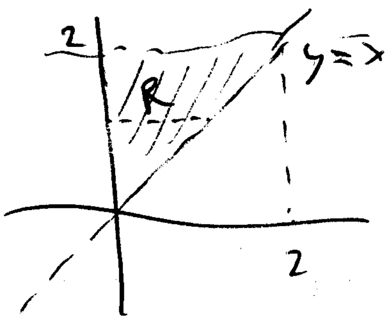
Show all work. How you get your answer is just as important, if not more important, than the answer itself. If you think it, write it!

1. Evaluate the iterated integral

$$\int_0^2 \int_x^2 x^2(y^4+1)^{1/3}\, dy\, dx$$

by rewriting the integral to reverse the order of integration.

(Note: the integral *cannot* be evaluated in the order given....)



$$= \iint_R x^2(y^4+1)^{1/3}\, dA$$

$$= \int_0^2 \int_0^y x^2(y^4+1)^{1/3}\, dx\, dy$$

$$= \int_0^2 \frac{x^3}{3}(y^4+1)^{1/3}\Big|_0^y\, dy = \int_0^2 \frac{y^3}{3}(y^4+1)^{1/3}\, dy$$

$u = y^4+1$

$du = 4y^3 dy$

$y^3 dy = \frac{1}{4} du$

$y=0 \quad u=1$

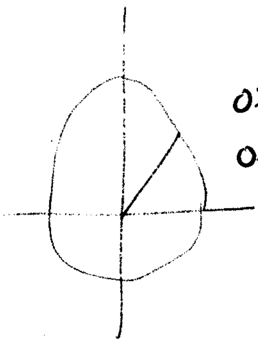
$y=2 \quad u=17$

$$= \frac{1}{12}\int_1^{17} u^{1/3}\, du = \frac{1}{12}\cdot\frac{3}{4}\, u^{4/3}\Big|_1^{17}$$

$$\boxed{= \frac{1}{16}\left(17^{4/3}-1\right)}$$

2. (20 pts.) Find the center of mass of the region R lying inside of the polar curve $r = 2 + \sin\theta$

$$\text{Area} = \iint_R 1\, dA = \int_0^{2\pi}\int_0^{2+\sin\theta} r\, dr\, d\theta$$



$0 \le r \le 2+\sin\theta$
$0 \le \theta \le 2\pi$

$$= \int_0^{2\pi} \frac{r^2}{2}\Big|_0^{2+\sin\theta} d\theta = \frac{1}{2}\int_0^{2\pi} (2+\sin\theta)^2\, d\theta$$

$$= \frac{1}{2}\int_0^{2\pi} 4 + 4\sin\theta + \sin^2\theta\, d\theta$$

$$= \frac{1}{2}\int_0^{2\pi} 4 + 4\sin\theta + \frac{1}{2}(1-\cos 2\theta)\, d\theta$$

$$= \frac{1}{2}\left(4\theta - 4\cos\theta + \frac{1}{2}\theta - \frac{1}{4}\sin 2\theta\right)\Big|_0^{2\pi}$$

$$= \frac{1}{2}\left((8\pi - 4 + \pi - 0) - (0 - 4 + 0 - 0)\right) = \frac{9\pi}{2}$$

$$\iint_R x\, dA = \int_0^{2\pi}\int_0^{2+\sin\theta} (r\cos\theta) r\, dr\, d\theta = \int_0^{2\pi}\int_0^{2+\sin\theta} r^2\cos\theta\, dr\, d\theta$$

$$= \int_0^{2\pi} \frac{r^3}{3}\cos\theta\Big|_0^{2+\sin\theta} d\theta = \frac{1}{3}\int_0^{2\pi} (2+\sin\theta)^3\cos\theta\, d\theta$$

$u = 2+\sin\theta$
$du = \cos\theta\, d\theta$

$$= \frac{1}{3}\int u^3\, du\Big|_{u=2+\sin\theta}\Big|_0^{2\pi} = \frac{1}{12}u^4\Big|_{u=2+\sin\theta}\Big|_0^{2\pi}$$

$\theta = 0 \quad u = 2$
$\theta = 2\pi \quad u = 2$

$$= \frac{1}{12}(2+\sin\theta)^4\Big|_0^{2\pi} = \frac{1}{12}(2^4 - 2^4) = 0$$

So $\bar{x} = \dfrac{0}{\frac{9\pi}{2}} = \boxed{0}$.

$$\iint_R y\,dA = \int_0^{2\pi}\int_0^{2+\sin\theta} (r\sin\theta)\,r\,dr\,d\theta$$

$$= \int_0^{2\pi}\int_0^{2+\sin\theta} r^2\sin\theta\,dr\,d\theta = \int_0^{2\pi} \frac{r^3}{3}\sin\theta\,\Big|_0^{2+\sin\theta} d\theta$$

$$= \frac{1}{3}\int_0^{2\pi} (2+\sin\theta)^3\sin\theta\,d\theta$$

$$= \frac{1}{3}\int_0^{2\pi} \left(8+12\sin\theta+6\sin^2\theta+\sin^3\theta\right)\sin\theta\,d\theta$$

$$= \frac{1}{3}\int_0^{2\pi} 8\sin\theta+12\sin^2\theta+6\sin^3\theta+\sin^4\theta\,d\theta$$

$$= \frac{1}{3}\int_0^{2\pi} 8\sin\theta\,d\theta + \frac{12}{3}\int_0^{2\pi}\frac{1}{2}(1-\cos 2\theta) + \frac{6}{3}\int_0^{2\pi}(1-\cos^2\theta)\sin\theta\,d\theta$$
$$+ \frac{1}{3}\int_0^{2\pi}\left(\frac{1}{2}(1-\cos 2\theta)\right)^2 d\theta$$

$$= -\frac{8}{3}\cos\theta\Big|_0^{2\pi} + 2\theta\Big|_0^{2\pi} - \frac{2}{2}\sin 2\theta\Big|_0^{2\pi} + \frac{6}{3}\int (1-u^2)(-du)\Big|_{u=\cos\theta}\Big|_0^{2\pi}$$
$$+ \frac{1}{12}\int_0^{2\pi} 1-2\cos 2\theta+\cos^2 2\theta\,d\theta$$

$$= -\frac{8}{3}(1-1) + 4\pi - 1(0-0) + 2\left(\frac{u^3}{3}-u\right)\Big|_{u=\cos\theta}\Big|_0^{2\pi} + \frac{1}{12}\theta\Big|_0^{2\pi} - \frac{1}{12}\sin 2\theta\Big|_0^{2\pi}$$
$$+ \frac{1}{12}\int_0^{2\pi}\frac{1}{2}(1+\cos 2\theta)\,d\theta$$

$$= 0 + 4\pi - 0 + 2\left(\frac{\cos^3\theta}{3}-\cos\theta\right)\Big|_0^{2\pi} + \frac{\pi}{6} - \frac{1}{12}(0-0) + \frac{1}{24}\theta\Big|_0^{2\pi} + \frac{1}{48}\sin 2\theta\Big|_0^{2\pi}$$

$$= 4\pi + 2\left(\left(\tfrac{1}{3}-1\right)-\left(\tfrac{1}{3}-1\right)\right) + \frac{\pi}{6} - 0 + \frac{\pi}{12} + \frac{1}{48}(0-0) = \frac{(48+2+1)\pi}{12}$$

$$= \frac{51\pi}{12} = \frac{17\pi}{4} \qquad \underline{\text{So}} \qquad \bar{y} = \frac{17\pi}{4}\Big/\frac{9\pi}{2} = \boxed{\frac{17}{18}}$$

3. (20 pts.) Find the surface area of the part of the graph of the cone
$z = f(x,y) = (x^2+y^2)^{1/2}$
which lies over the region R in the plane lying between the graphs of
$y = x^4$ and $y = x^2$.

$$f(x,y) = (x^2+y^2)^{1/2} \qquad f_x = \tfrac{1}{2}(x^2+y^2)^{-1/2}(2x) = \frac{x}{(x^2+y^2)^{1/2}}$$

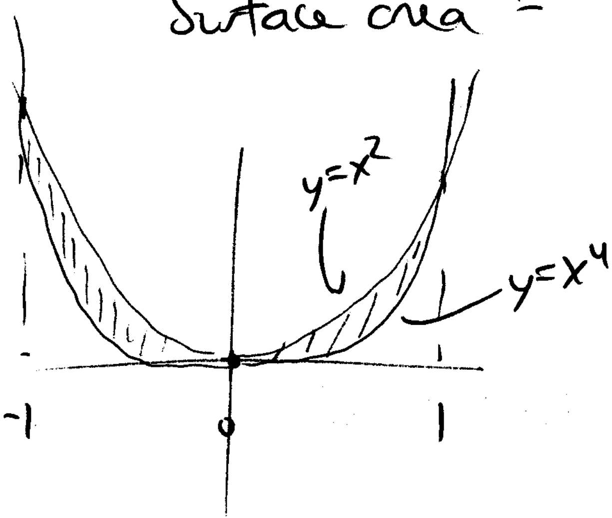
$$f_y = \tfrac{1}{2}(x^2+y^2)^{-1/2}(2y) = \frac{y}{(x^2+y^2)^{1/2}}$$

$$(f_x)^2+(f_y)^2+1 = \frac{x^2}{x^2+y^2} + \frac{y^2}{x^2+y^2} + 1$$

$$= \frac{x^2+y^2}{x^2+y^2} + 1 = 1+1 = 2 \ !$$

So

$$\text{Surface area} = \iint_R \sqrt{2}\, dA = \int_{-1}^{1}\int_{x^4}^{x^2} \sqrt{2}\, \cancel{dA}\, dy\, dx$$



$$= \int_{-1}^{1} \sqrt{2}\, y \Big|_{x^4}^{x^2} dx$$

$$= \int_{-1}^{1} \sqrt{2}x^2 - \sqrt{2}x^4\, dx = \frac{\sqrt{2}}{3}x^3 - \frac{\sqrt{2}}{5}x^5 \Big|_{-1}^{1}$$

$$= \left(\frac{\sqrt{2}}{3} - \frac{\sqrt{2}}{5}\right) - \left(-\frac{\sqrt{2}}{3} + \frac{\sqrt{2}}{5}\right)$$

$$= 2\sqrt{2}\left(\frac{1}{3} - \frac{1}{5}\right) = 2\sqrt{2}\left(\frac{5-3}{15}\right) = 2\sqrt{2}\left(\frac{2}{15}\right)$$

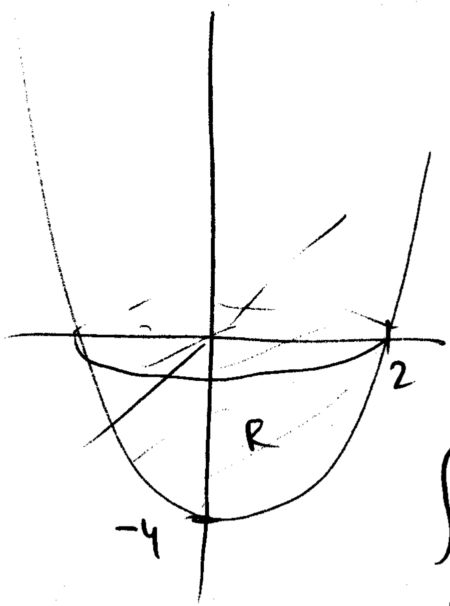
$$= \boxed{\frac{4\sqrt{2}}{15}}$$

If you did $\int_0^1 \int_{x^4}^{x^2} \sqrt{2}\, dy\, dx$, that's fine too....

4. (20 pts.) Find the integral of the function

$f(x, y, z) = x + y + z$

over the region lying between the graph of $z = x^2 + y^2 - 4$ and the x-y plane.



cylindrical!

$z = x^2+y^2-4 = r^2-4$ over disk of radius 2

$r^2-4 \le z \le 0$

$0 \le \theta \le 2\pi$

$0 \le r \le 2$

$$\iiint_R f\,dV = \int_0^2\int_0^{2\pi}\int_{r^2-4}^{0} (r\cos\theta + r\sin\theta + z)\, r\, dz\, d\theta\, dr$$

$$= \int_0^2\int_0^{2\pi}\int_{r^2-4}^{0} r^2\cos\theta + r^2\sin\theta + rz\, dz\, d\theta\, dr$$

$$= \int_0^2\int_0^{2\pi} r^2 z\cos\theta + r^2 z\sin\theta + r\frac{z^2}{2}\Big|_{r^2-4}^{0} d\theta\, dr$$

$$= \int_0^2\int_0^{2\pi} (0+0+0) - \left(r^2(r^2-4)\cos\theta + r^2(r^2-4)\sin\theta + r\frac{(r^2-4)^2}{2}\right) d\theta\, dr$$

$$= -\int_0^2 r^2(r^2-4)\sin\theta - r^2(r^2-4)\cos\theta + \frac{1}{2}r(r^2-4)^2\theta\Big|_0^{2\pi} dr$$

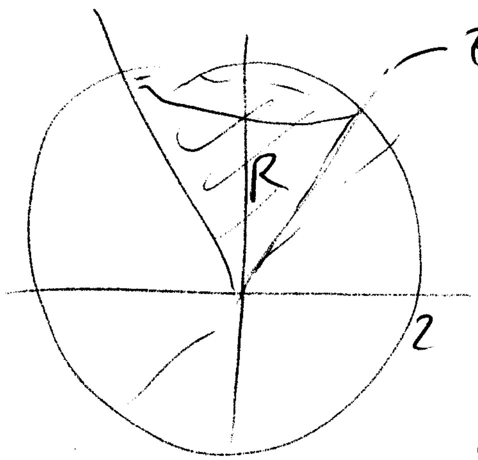
$$= -\int_0^2 \left(0 - \cancel{r^2(r^2-4)} + \pi r(r^2-4)^2\right) - \left(0 - \cancel{r^2(r^2-4)} + 0\right)\, dr$$

$$= -\pi\int_0^2 r(r^2-4)^2\, dr \underset{\substack{u = r^2-4 \\ du = 2r\,dr}}{=} -\pi\int_{-4}^{0} \frac{1}{2}u^2\, du = -\pi\frac{1}{6}u^3\Big|_{-4}^{0} = \boxed{-\frac{\pi}{6}(4)^3}$$

5. (20 pts.) Use spherical coordinates to find the volume of the region lying inside of the sphere of radius 2 and above the cone

$z = \sqrt{3}\sqrt{x^2+y^2}$

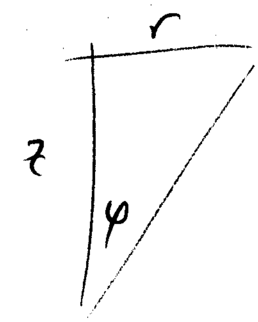
(Hint: in spherical coordinates, a cone is $\varphi =$ constant ; which constant?)

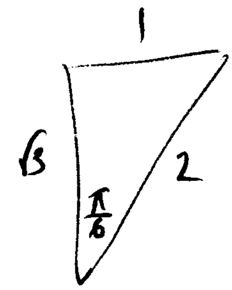


$z = \sqrt{3}\sqrt{x^2+y^2} = \sqrt{3}\, r$

$\frac{r}{z} = \frac{z/\sqrt{3}}{z} = \tan\varphi = \frac{1}{\sqrt{3}}$

$\varphi = \frac{\pi}{6}$!





$$R: \quad \begin{array}{l} 0 \le \varphi \le \frac{\pi}{6} \\ 0 \le \rho \le 2 \\ 0 \le \theta \le 2\pi \end{array}$$

$$\text{Volume} = \iiint_R 1\, dV = \int_0^{2\pi}\int_0^2\int_0^{\frac{\pi}{6}} \rho^2 \sin\varphi \, d\varphi\, d\rho\, d\theta$$

$$= \int_0^{2\pi}\int_0^2 -\rho^2\cos\varphi \Big|_0^{\frac{\pi}{6}} d\rho\, d\theta = \int_0^{2\pi}\int_0^2 -\rho^2\left(\cos\tfrac{\pi}{6} - \cos 0\right) d\rho\, d\theta$$

$$= \int_0^{2\pi}\int_0^2 \rho^2\left(1-\tfrac{\sqrt{3}}{2}\right) d\rho\, d\theta = \left(1-\tfrac{\sqrt{3}}{2}\right)\int_0^{2\pi} \frac{\rho^3}{3}\Big|_0^2 d\theta$$

$$= \left(1-\tfrac{\sqrt{3}}{2}\right)\left(\tfrac{8}{3}\right)\int_0^{2\pi} d\theta = \left(1-\tfrac{\sqrt{3}}{2}\right)\left(\tfrac{8}{3}\right)(2\pi)$$

$$= \frac{(2-\sqrt{3})(8)(2\pi)}{2\cdot 3} = \boxed{\frac{8\pi}{3}(2-\sqrt{3})}$$