Math 208H, Section 1

Quiz number 1 Solutions

Show all work. How you get your answer is just as important, if not more important, than the answer itself. If you think it, write it!

Find an equation of the plane that passes through the points

$$P = (1,2,2)\ ,\ Q = (3,1,-1),\ \text{and}\ R = (2,3,5)$$

We need the normal vector, so we compute:
$\vec{PQ} = \langle 3-1, 1-2, -1-2\rangle = \langle 2,-1,-3\rangle$ and
$\vec{PR} = \langle 2-1, 3-2, 5-2\rangle = \langle 1,1,3\rangle$.
So $\vec{N} = \langle 2,-1,-3\rangle \times \langle 1,1,3\rangle = \begin{vmatrix} -1 & -3 \\ 1 & 3 \end{vmatrix} \vec{i} - \begin{vmatrix} 2 & -3 \\ 1 & 3 \end{vmatrix} \vec{j} + \begin{vmatrix} 2 & -1 \\ 1 & 1 \end{vmatrix} \vec{k}$
$= (-3-(-3)\vec{i} - (6-(-3))\vec{j} + (2-(-1))\vec{k} = \langle 0,-9,3\rangle$.

[Quick check: $\vec{N}$ is $\perp$ to $\vec{PQ}$ and $\vec{PR}$...]

Then our equation for the plane can be given by
$\vec{X} \bullet \vec{N} = \vec{P} \bullet \vec{N}$, i.e., $\langle x,y,z\rangle \bullet \langle 0,-9,3\rangle = \langle 1,2,2\rangle \bullet \langle 0,-9,3\rangle$, i.e.,
$0x - 9y + 3z = 1\cdot 0 + 2\cdot(-9) + 3\cdot 2 = 0 - 18 + 6 = -12$, i.e.,
$-9y + 3z = -12$ [or, removing common factors, $3y - z = 4$]

[Second quick check: our three points P,Q,R lie on this plane (i.e., satisfy this equation!)...]

Does the point (3,2,1) lie on this plane??

This is really asking: does $(x,y,z) = (3,2,1)$ satsify $3y - z = 4$?
Since $3(2) - (1) = 6 - 1 = 5 \neq 4$, the answer is "NO!!" .

[N.B.: There were many different (correct) ways to find the answers to these questions, by choosing a different point (Q or R) and/or vectors pointing in the plane. The resulting normal vector $\vec{N}$ would be some (scalar) multiple of the one above, though!]

Name:

Math 208H, Section 1

Quiz number 2

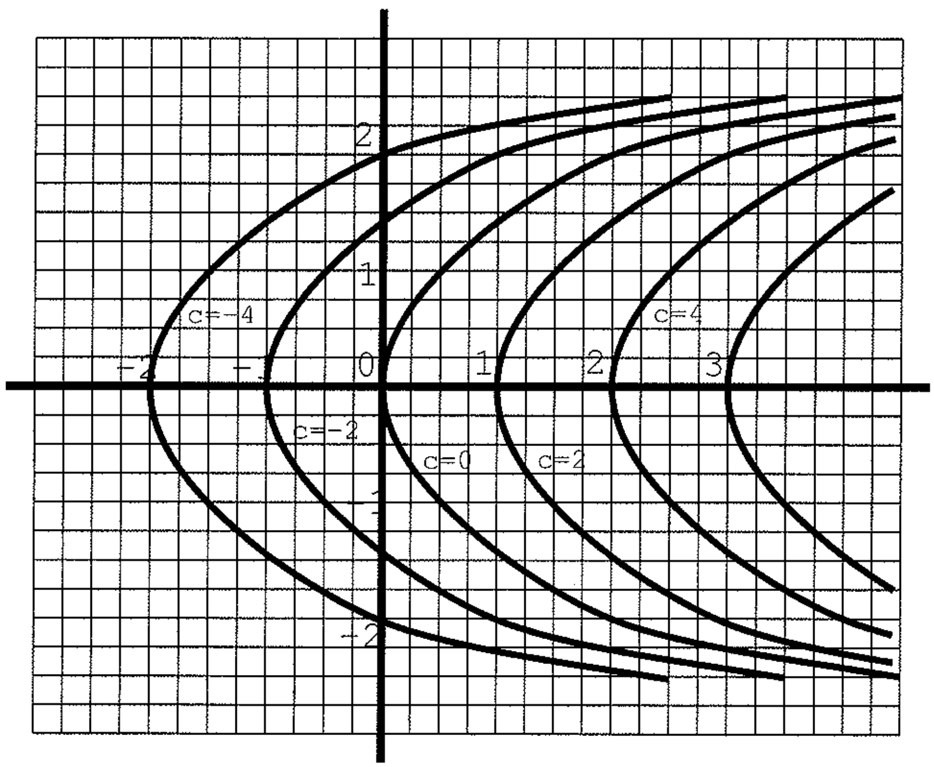
Show all work. How you get your answer is just as important, if not more important, than the answer itself. If you think it, write it!

1. For the function

$$z = f(x,y) = 2x - y^2$$

sketch the **level curves** for the values $z = -1$, $z = 0$, and $z = 1$, on the grid below. Be sure to label everything that you feel needs labelling. (Hint: thinking of x as a function of y might work better than the other way around...)

For level curves we look at points where $f(x,y) = c =$ constant, so we have $2x - y^2 = c$, so $2x = y^2 + c$, so $x = \frac{1}{2}y^2 + \frac{c}{2}$. These look like parabolas (somewhat wider the our usual one), but with roles of x and y reversed, so they open to the right rather than up. For convenience the graph below plots the level curves passing through $y = 0$ and $x = -2, -1, 0, 1, 2, 3$, so the corresponding values of c are $c = -4, -2, 0, 2, 4, 6$.



Math 208H, Section 1

Quiz number 3 Solutions

Find the partial derivatives of the function

$$z = f(x,y) = \frac{xy}{x^2+y^2}$$

The function is, essentially, a quotient, so we apply the quotient rule!

$$\frac{\partial f}{\partial x} = \frac{[\frac{\partial}{\partial x}(xy)][x^2+y^2] - [\frac{\partial}{\partial x}(x^2+y^2)][xy]}{(x^2+y^2)^2} = \frac{(y)(x^2+y^2)-(2x)(xy)}{(x^2+y^2)^2}$$

$$= \frac{x^2y+y^3-2x^2y}{(x^2+y^2)^2} = \frac{y^3-x^2y}{(x^2+y^2)^2}$$

$$\frac{\partial f}{\partial y} = \frac{[\frac{\partial}{\partial y}(xy)][x^2+y^2] - [\frac{\partial}{\partial y}(x^2+y^2)][xy]}{(x^2+y^2)^2} = \frac{(x)(x^2+y^2)-(2y)(xy)}{(x^2+y^2)^2}$$

$$= \frac{x^3+xy^2-2xy^2}{(x^2+y^2)^2} = \frac{x^3-xy^2}{(x^2+y^2)^2}$$

Use these to find the rate of change of the composite function

$$F(t) = f(t^2+1, \cos t) = f(x(t), y(t))$$

at $t = 0$.

We know, from the chain rule, that $\displaystyle\frac{df}{dt} = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt}$.

Since $\displaystyle\frac{dx}{dt} = 2t$ and $\displaystyle\frac{dy}{dt} = -\sin t$, which at $t = 0$ are both equal to 0, and $x(0) = 1 = y(0)$, we find that

$$\frac{df}{dt}|_{t=0} = \frac{\partial f}{\partial x}|_{(x,y)=(1,1)}\frac{dx}{dt}|_{t=0} + \frac{\partial f}{\partial y}|_{(x,y)=(1,1)}\frac{dy}{dt}|_{t=0}$$

$$= \frac{1^3-1^2\cdot 1}{(1^2+1^2)^2}(0) + \frac{1^3-1\cdot 1^2}{(1^2+1^2)^2}(0) = \frac{0}{4}(0) + \frac{0}{4}(0) = 0\cdot 0 + 0\cdot 0 = 0+0 = 0$$

Math 208H, Section 1

Quiz number 4 Solutions

Show all work. How you get your answer is just as important, if not more important, than the answer itself. If you think it, write it!

In which direction is the function

$$z = f(x,y) = \frac{xy^2}{x+y}$$

increasing the fastest, at the point $(a,b) = (1,2)$? What is the equation for the tangent plane to the graph of f, at this same point?

A function increases fastest in the direction of its gradient, ∇f. So we compute:

$$\frac{\partial f}{\partial x} = \frac{(y^2)(x+y)-(1)(xy^2)}{(x+y)^2} = \frac{y^3}{(x+y)^2}$$

$$\frac{\partial f}{\partial y} = \frac{(2xy)(x+y)-(1)(xy^2)}{(x+y)^2} = \frac{2x^2y+xy^2}{(x+y)^2}$$

At $(1,2)$, we have $\nabla f = \langle \frac{2^3}{(1+2)^2}, \frac{2\cdot 1^2\cdot 2+1\cdot 2^2}{(1+2)^2}\rangle = \langle \frac{8}{9}, \frac{8}{9}\rangle$

[If you prefer a unit vector, $|\nabla f| = \frac{8}{9}\sqrt{2}$, so $\frac{\nabla f}{|\nabla f|} = \langle \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\rangle$.]

So f increases fastest in the direction of $\nabla f = \langle \frac{8}{9}, \frac{8}{9}\rangle$, at the point $(1,2)$.

For the tangent plane, we have $f(1,2) = \frac{1\cdot 2^2}{(1+2)} = \frac{4}{3}$, and so the equation for the plane tangent to the graph of f at $(1,2,\frac{4}{3})$ is

$$\langle \frac{8}{9}, \frac{8}{9}, -1\rangle \circ \langle x-1, y-2, z-\frac{4}{3}\rangle = 0, \text{ or}$$

$$\frac{8}{9}(x-1) + \frac{8}{9}(y-2) - (z-\frac{4}{3}) = 0, \text{ or}$$

$$z = \frac{8}{9}x + \frac{8}{9}y - \frac{4}{3}$$

Math 208H, Section 1

Quiz number 5 Solution

Show all work. How you get your answer is just as important, if not more important, than the answer itself. If you think it, write it!

Find the **minimum** value of the function

$$f(x,y) = x^2y$$

subject to the constraint

$$g(x,y) = x^2 + y^2 = 6\ .$$

We compute $\nabla f = (2xy, x^2)$ and $\nabla g = (2x, 2y)$, so we wish to solve

$$2xy = 2\lambda x\ ,\ x^2 = 2\lambda y\ ,\ \text{and}\ x^2 + y^2 = 6$$

The first equation gives $2x(y-\lambda) = 0$, so either $x = 0$ or $y = \lambda$.

$x = 0$ implies, together with $x^2 + y^2 = 6$, that $y^2 = 6$, so $y = \sqrt{6}$ or $y = -\sqrt{6}$. This gives us the possible points $(0, \sqrt{6})$ and $(0, -\sqrt{6})$.

$y = \lambda$ implies, using $x^2 = 2\lambda y$, that $x^2 = 2\lambda^2$, so $x = \pm\sqrt{2}\lambda$. Then $x^2 + y^2 = 6$ implies that $(\pm\sqrt{2}\lambda)^2 + \lambda^2 = 3\lambda^2 = 6$, so $\lambda^2 = 2$, so $\lambda = \pm\sqrt{2}$. This in turn implies that $x = \sqrt{2}\lambda = \pm 2$.

This gives us the possible points $(2, \sqrt{2})$, $(-2, \sqrt{2})$, $(2, -\sqrt{2})$, and $(-2, -\sqrt{2})$.

Plugging all of these points into f gives the values $0, 0, 4\sqrt{2}, -4\sqrt{2}, -4\sqrt{2}$, and $4\sqrt{2}$. The leftmost of these values is $-4\sqrt{2}$, so this is the minimum value of f subject to our constraint.

Name: Solutions

Math 208H, Section 1

Quiz number 6

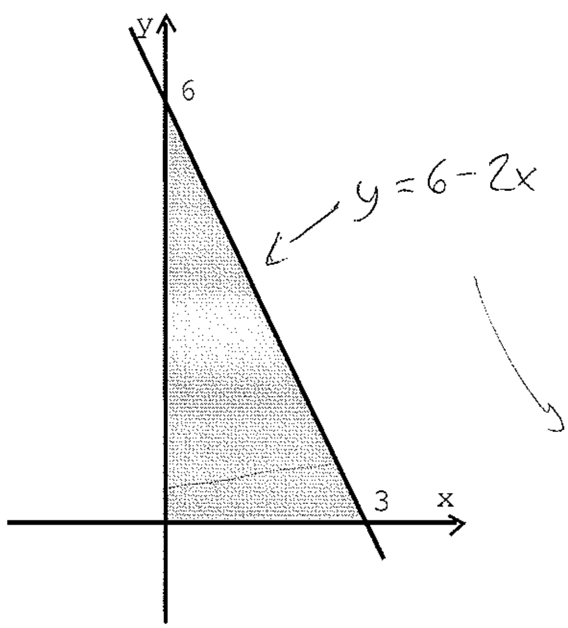
Show all work. How you get your answer is just as important, if not more important, than the answer itself. If you think it, write it!

Show how to set up the following double integral as an iterated integral in **both** of our 'standard' ways, but do not evaluate the resulting integrals! The function is

$$f(x, y) = xy^2,$$

over the region lying in the first quadrant of the x-y plane and lying below the line passing through the points $(3, 0)$ and $(0, 6)$.

(see figure).



$0 \le y \le 6-2x$

for $0 \le x \le 3$

$$\int_0^3 \int_0^{6-2x} xy^2 \, dy \, dx$$

$y+2x=6 \quad 2x = 6-y \quad x = \frac{6-y}{2}$

$0 \le x \le \frac{6-y}{2}$

for $0 \le y \le 6$

$$\int_0^6 \int_0^{\frac{6-y}{2}} xy^2 \, dx \, dy$$

Name: Solutions

Math 208H, Section 1

Quiz number 7

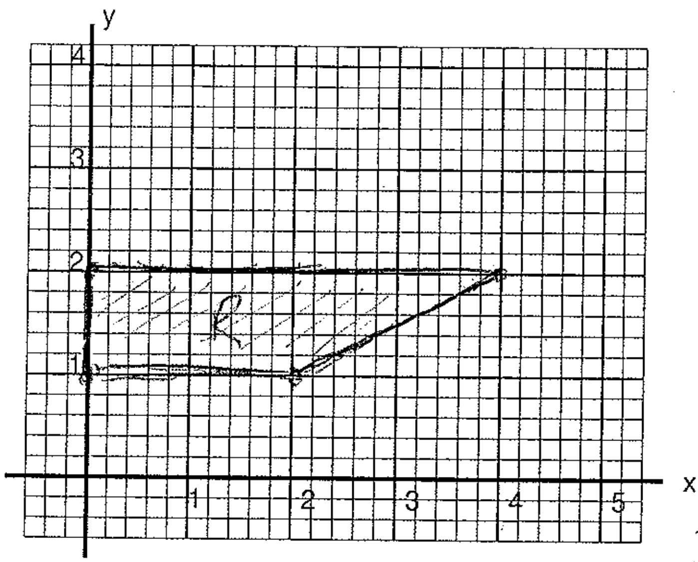
Show all work. How you get your answer is just as important, if not more important, than the answer itself. If you think it, write it!

Under the change of variables

$$x = uv \quad , \quad y = u$$

sketch the region R in the xy-plane that the rectangle S in the uv-plane given by $1 \le u \le 2 \quad , \quad 0 \le v \le 2$ gets sent to. [Hint: look at where the vertical and horizontal boundary line segments are sent. Hint for the hint: where are the corners of the rectangle sent?]

Use this change of variables to **convert** the integral $\displaystyle\int\int_R x + y\, dA$ into an integral over S. [Extra credit: evaluate the resulting integral!]



$(u,v) = (1,0) \longrightarrow (x,y) = (0,1)$

$(u,v) = (1,2) \longrightarrow (x,y) = (2,1)$

$(u,v) = (2,0) \longrightarrow (x,y) = (0,2)$

$(u,v) = (2,2) \longrightarrow (x,y) = (4,2)$

$f(x,y) = x+y = uv + u$

$x_u = v,\ x_v = u,\ y_u = 1,\ y_v = 0$

$J = |x_u y_v - x_v y_u| = |v \cdot 0 - u \cdot 1| = |-u| = u$

So $\displaystyle\iint_R x+y\, dA = \int_0^2 \int_1^2 (uv+u)\, u \, du\, dv$

$\displaystyle = \int_0^2 \int_1^2 u^2 v + u^2\, du\, dv = \int_0^2 \left(\frac{u^3}{3} v + \frac{u^3}{3} \right)\Big|_{u=1}^{u=2} dv = \int_0^2 \left(\frac{8}{3}v + \frac{8}{3}\right) - \left(\frac{1}{3}v + \frac{1}{3}\right) dv$

$\displaystyle = \int_0^2 \frac{7}{3}v + \frac{7}{3}\, dv = \frac{7}{6}v^2 + \frac{7}{3}v \Big|_0^2 = \frac{7}{6}(4) + \frac{7}{3}(2) = \frac{14}{3} + \frac{14}{3} = \boxed{\frac{28}{3}}$ or:

$\displaystyle = \int_1^2 \int_0^2 u^2 v + u^2\, dv\, du = \int_1^2 \frac{u^2 v^2}{2} + u^2 v \Big|_0^2 du = \int_1^2 2u^2 + 2u^2\, du = \int_1^2 4u^2\, du = \frac{4}{3}u^3 \Big|_1^2$

$\displaystyle = \frac{4}{3}(8-1) = \frac{4}{3}(7) = \boxed{\frac{28}{3}}$ (!)

Name: Solutions

Math 208H, Section 1

Quiz number 8

Show all work. How you get your answer is just as important, if not more important, than the answer itself. If you think it, write it!

The region W pictured below consists of that part of the inside of the sphere of radius 3 which lies in the first octant in 3-space.

Set up **but do not compute** the integral of the function $f(x, y, z) = x + y$ over the region W, using

(a) cylindrical coordinates

(b) spherical coordinates .

cylindrical:

$x = r\cos\theta$
$y = r\sin\theta$
$z = z$
$J = r$

spherical:

$x = \rho\cos\theta\sin\varphi$
$y = \rho\sin\theta\sin\varphi$
$z = \rho\cos\varphi$
$J = \rho^2\sin\varphi$

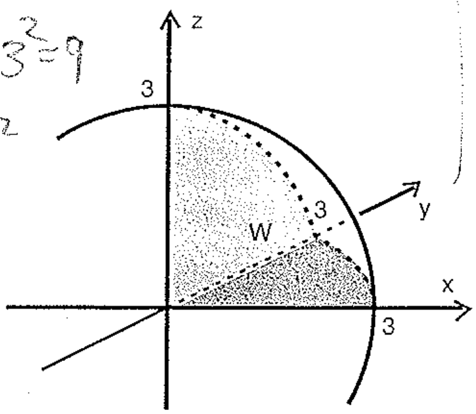
$x^2+y^2+z^2 = 3^2 = 9$

$z^2 = 9 - x^2 - y^2 = 9 - r^2$

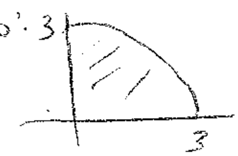
$z = \sqrt{9-r^2}$

or

$\rho = 3$



x-y shadow:



$0 \le r \le 3$
$0 \le \theta \le \frac{\pi}{2}$

cylindrical: W is

$0 \le z \le \sqrt{9-r^2}$

for $0 \le r \le 3$
$0 \le \theta \le \frac{\pi}{2}$

spherical: W is

$0 \le \rho \le 3$
$0 \le \theta \le \frac{\pi}{2}$
$0 \le \varphi \le \frac{\pi}{2}$

integral:

$$\int_0^{\frac{\pi}{2}}\int_0^{3}\int_0^{\sqrt{9-r^2}} (r\cos\theta + r\sin\theta)\, r \; dz\, dr\, d\theta$$

integral:

$$\int_0^{\frac{\pi}{2}}\int_0^{\frac{\pi}{2}}\int_0^{3} (\rho\cos\theta\sin\varphi + \rho\sin\theta\sin\varphi)\cdot \rho^2\sin\varphi \; d\rho\, d\theta\, d\varphi$$

Name: Solution

Math 208H, Section 1

Quiz number 9

Show all work. How you get your answer is just as important, if not more important, than the answer itself. If you think it, write it!

Compute the line integral $\int_\gamma \vec{F} \bullet d\vec{r}$, where

$$\vec{F}(x,y) = (xy, x+y) \text{ and}$$
$$\gamma(t) = (\cos t, \sin t) \text{ for } 0 \le t \le \pi \ .$$

$\gamma'(t) = (-\sin t, \cos t)$ $\quad F(\gamma(t)) = (\cos t \sin t, \cos t + \sin t)$

$$F(\gamma(t)) \cdot \gamma'(t) = (\cos t \sin t, \cos t + \sin t) \cdot (-\sin t, \cos t)$$
$$= -\sin^2 t \cos t + \cos^2 t + \sin t \cos t$$

$$\int_\gamma \vec{F} \cdot d\vec{r} = \int_0^\pi \vec{F}(\gamma(t)) \cdot \gamma'(t)\, dt$$
$$= \int_0^\pi -\sin^2 t \cos t + \cos^2 t + \sin t \cos t \; dt$$
$$= \int_0^\pi -\sin^2 t \cos t \, dt + \int_0^\pi \cos^2 t \, dt + \int_0^\pi \sin t \cos t \, dt$$
$$= \int_0^\pi (\sin t - \sin^2 t) \cos t \, dt + \int_0^\pi \frac{1}{2}(1 + \cos 2t)\, dt$$

$u = \sin t \quad du = \cos t \, dt$

$t = 0 \to u = 0 \quad t = \pi \to u = 0$

$$= \int_0^0 u - u^2 \, du + \frac{1}{2}\left(t + \frac{1}{2}\sin 2t\right)\Big|_0^\pi$$
$$= \frac{u^2}{2} - \frac{u^3}{3}\Big|_0^0 + \frac{1}{2}\left(t + \frac{1}{2}\sin(2t)\right)\Big|_0^\pi$$
$$= [(0-0)-(0-0)] + \frac{1}{2}\left[\left(\pi + \frac{1}{2}\cdot 0\right) - \left(0 + \frac{1}{2}\cdot 0\right)\right]$$
$$= 0 + \frac{1}{2}(\pi) = \boxed{\frac{\pi}{2}}$$