Math 208H

A formula for the area of a polygon

We can use Green's Theorem to find a formula for the area of a polygon P in the plane with corners at the points

$$(x_1, y_1), (x_2, y_2), \ldots, (x_n, y_n) \text{ (reading counterclockwise around } P).$$

The idea is to use the formulas (derived from Green's Theorem)

$$\text{Area inside } P = \int_P \langle 0, x\rangle \cdot\ dr = \int_P \langle -y, 0\rangle \cdot\ dr$$

Each side of the polygon P can be parametrized as a straight line segment P_i by

$$r_i(t) = (x_i + t(x_{i+1} - x_i), y_i + t(y_{i+1} - y_i))\ ,\ 0 \le t \le 1$$

for $i = 1, \ldots, n$ (where $x_{n+1} = x_1$ is thought to have returned to the beginning of the polygon P, and similarly $y_{n+1} = y_1$). Then

$$\text{Area} = \int_P \langle 0, x\rangle \cdot\ dr = \int_{P_1} \langle 0, x\rangle \cdot\ dr + \cdots + \int_{P_n} \langle 0, x\rangle \cdot\ dr$$

(and similarly for the other integral). Then we can compute:

$r_i'(t) = \langle x_{i+1} - x_i, y_{i+1} - y_i\rangle$, so

$$\begin{aligned}
&\int_{P_i} \langle 0, x\rangle \cdot\ dr \\
&\qquad = \int_0^1 \langle 0, x_i + t(x_{i+1} - x_i)\rangle \cdot \langle x_{i+1} - x_i, y_{i+1} - y_i\rangle\ dt \\
&\qquad = (y_{i+1} - y_i) \int_0^1 x_i + t(x_{i+1} - x_i)\ dt \\
&\qquad = (y_{i+1} - y_i)(x_i t + \frac{1}{2}t^2(x_{i+1} - x_i))|_0^1 \\
&\qquad = (y_{i+1} - y_i)(x_i + \frac{1}{2}(x_{i+1} - x_i)) \\
&\qquad = \frac{1}{2}(y_{i+1} - y_i)(x_i + x_{i+1})
\end{aligned}$$

Summing over i, we get our formula!

$$\text{Area} = \frac{1}{2}(y_2 - y_1)(x_1 + x_2) + \cdots + \frac{1}{2}(y_n - y_{n-1})(x_{n-1} + x_n) + \frac{1}{2}(y_1 - y_n)(x_1 + x_n)$$

This formula seems to treat the x's and the y's differently; one is in a sum, the other a difference. We can get a *better* formula if we *also* compute the area as

$$\text{Area} = \int_P \langle -y, 0\rangle \cdot\ dr = \int_{P_1} \langle -y, 0\rangle \cdot\ dr + \cdots + \int_{P_n} \langle -y, 0\rangle \cdot\ dr$$

Using the exact same parametrizations for the sides, we can compute
[i.e., *you* should compute]:

$\int_{P_i} \langle -y, 0\rangle \cdot\ dr = \frac{1}{2}(x_i - x_{i+1})(y_i + y_{i+1})$, so

$$\text{Area} = \frac{1}{2}(x_1 - x_2)(y_1 + y_2) + \cdots + \frac{1}{2}(x_{n-1} - x_n)(y_{n-1} + y_n) + \frac{1}{2}(x_n - x_1)(y_1 + y_n)$$

But! Since both formulas compute the same number (the area), their *average* does, as well. But!

$$
\begin{aligned}
&\frac{1}{2}(\frac{1}{2}(y_{i+1}-y_i)(x_i+x_{i+1})+\frac{1}{2}(x_i-x_{i+1})(y_i+y_{i+1})) \\
&=\frac{1}{4}((y_{i+1}-y_i)(x_i+x_{i+1})+(x_i-x_{i+1})(y_i+y_{i+1})) \\
&=\frac{1}{4}(y_{i+1}x_i-y_ix_i+y_{i+1}x_{i+1}-y_ix_{i+1}+x_iy_i-x_{i+1}y_i+x_iy_{i+1}-x_{i+1}y_{i+1}) \\
&=\frac{1}{4}(y_{i+1}x_i+x_iy_{i+1}-y_ix_i+x_iy_i+y_{i+1}x_{i+1}-x_{i+1}y_{i+1}-y_ix_{i+1}-x_{i+1}y_i) \\
&=\frac{1}{4}(2x_iy_{i+1}-2x_{i+1}y_i) \\
&=\frac{1}{2}(x_iy_{i+1}-x_{i+1}y_i) \\
&=\frac{1}{2}\begin{vmatrix} x_i & x_{i+1} \\ y_i & y_{i+1} \end{vmatrix}
\end{aligned}
$$

So when we sum over these quantities, we get

$$
\begin{aligned}
\text{Area} &= \frac{1}{2}[(x_1y_2-x_2y_1)+\cdots(x_{n-1}y_n-x_ny_{n-1})+(x_ny_1-x_1y_n)] \\
&= \frac{1}{2}\left(\begin{vmatrix} x_1 & x_2 \\ y_1 & y_2 \end{vmatrix}+\cdots+\begin{vmatrix} x_{n-1} & x_n \\ y_{n-1} & y_n \end{vmatrix}+\begin{vmatrix} x_n & x_1 \\ y_n & y_1 \end{vmatrix}\right)
\end{aligned}
$$

This formula "feels" better (doesn't it?); it treats the x-coordinates and y-coordinates more equally. The two intermediate formulas are more lop-sided in that regard....

Hindsight is 20-20, they say; can you *explain* this formula differently, now that we have discovered it? A hint: each of the terms in the sum can be interpreted as the area of a triangle, with two sides equal to a certain pair of vectors....