Math 208H

Divergence-free vector fields are curls of things

We know that the curl of a vector field is a vector field which is divergence-free:

$$\text{div}(\text{curl}(\vec{F})) = 0$$

We wish to demonstrate that the reverse is also true. Suppose $\vec{F} = \langle P, Q, R\rangle$ is a vector field that is divergence-free, i.e.,

$$\text{div}\vec{F} = P_x + Q_y + R_z = 0$$

We want to show that there is a vector field $\vec{G} = \langle S, T, U\rangle$ with

$$\text{curl}\vec{G} = \langle U_y - T_z, S_z - U_x, T_x - S_y\rangle = \vec{F}$$

The basic little trick that makes it possible to show this is the fact that for any function $f(x, y, z)$, $\text{curl}(\nabla f) = 0$; this is really the statement that mixed partial derivatives are equal ($f_{xy} = f_{yx}$, $f_{xz} = f_{zx}$, $f_{yz} = f_{zy}$).

So for any vector field $\vec{G}$ and any function f, $\text{curl}\vec{G} = \text{curl}(\vec{G} + \nabla f)$, i.e, we can change the vector field $\vec{G}$ in a controllable way *without* changing its curl.

This allows us to simplify the task of finding $\vec{G}$ by *first* choosing a function f with $f_z = -U$ (e.g., integrate $-U$, dz !), so

$G + \nabla f = \langle S + f_x, T + f_y, U + f_z\rangle = \langle S + f_x, T + f_y, 0\rangle$

and this has the same curl as $\vec{G}$. This means that to find the $\vec{G}$ we want, we only need to look at vector fields with third coordinate 0 ! (In fact, we could make any one coordinate equal to 0, by a similar argument.)

So our problem now becomes to find, if $P_x + Q_y + R_z = 0$, a vector field $\vec{G} = \langle S, T, 0\rangle$ with $\text{curl}\vec{G} = \langle -T_z, S_z, T_x - S_y\rangle = \langle P, Q, R\rangle$. That is, we want

$$T_z = -P \ , \ S_z = Q \ , \text{ and } T_x - S_y = R$$

So, we do the only thing we can! We integrate $-P$, dz , to get T, and Q , dz , to get S . *But* there are constants of integration involved, as well...

$T = -\int P\ dz\ + C_1(x, y)$ (for some function C_1)

$S = \int Q\ dz$ (there is another constant of integration here, too, but we won't *need* it)

We can figure out what $C_1(x, y)$ *should be* by using our last condition:

(*) $R = T_x - S_y = -(\int P\ dz + C_1)_x - (\int Q\ dz)_y = -(\int P\ dz)_x + (C_1)_x - (\int Q\ dz)_y$,

so

(**) $(C_1)_x = R + (\int P\ dz)_x + (\int Q\ dz)_y$;

integrating the right-hand side of this equation, dx, gives us C_1 ,
and therefore gives us T (and S).

But where did we *use* that fact that $P_x + Q_y + R_z = 0$?! It was at (*); i.e, it insures that there *is* a function C_1 of x and y for which (*) is true. This is because in (**)

$((C_1)_x)_z = (R + (\int P\ dz)_x + (\int Q\ dz)_y)_z = R_z + (\int P\ dz)_{xz} + (\int Q\ dz)_{yz} =$
$R_z + (\int P\ dz)_{zx} + (\int Q\ dz)_{zy} = R_z + ((\int P\ dz)_z)_x + ((\int Q\ dz)_z)_y = R_z + P_x + Q_y = 0$

So $(C_1)_x$ (and therefore C_1) , as defined by (*), doesn't depend on z, i.e., it *is* only a function of x and y !