**Math 208H**
**Why Green's Theorem is true**

Let $R =$ a region in the plane, and $C = \partial R =$ the boundary of $R$, traversed counterclockwise.

Let $F = \langle F_1, F_2 \rangle =$ a vector field on $R$, and let $\text{curl}(F) = (F_2)_x - (F_1)_y$

Then Green's Theorem says that

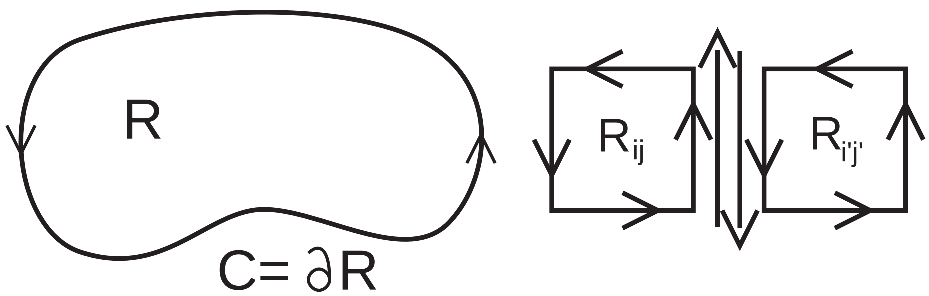
$$(*) \qquad \int\int_R \text{curl}(F)\ dA = \int_C F \cdot\ dr$$

To show this, we think of $R$ as being cut up into (or approximated by) a huge number of tiny rectangles $R_{ij}$.

Then (**) $\quad \int\int_R \text{curl}(F)\ dA = \Sigma_{i,j} \int\int_{R_{ij}} \text{curl}(F)\ dA$ , since $R$ is a "sum" of the $R_{ij}$'s .

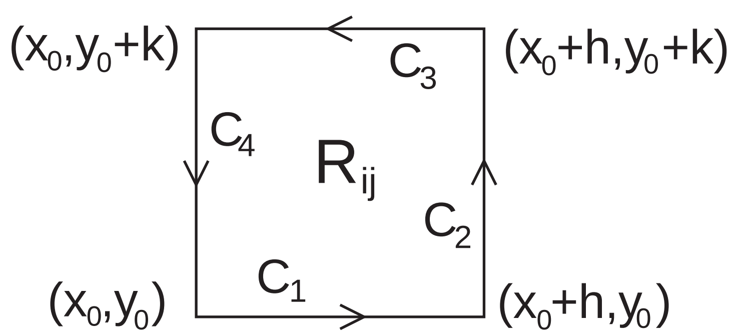
On the other hand, (***) $\quad \int_C F \cdot\ dr = \Sigma_{i,j} \int_{\partial R_{ij}} F \cdot\ dr$ ,

since the parts of the $\partial R_{ij}$ that lie *inside* of $R$ are counted twice in this sum, but are traversed in *opposite directions* when they appear. So all of the integrals over the pieces of the $\partial R_{ij}$ cancel, *except* over the parts that traverse $\partial R$ (which only get counted once!).




Because of these two equations (**) and (***), to verify (*) it is enough to show that
$\int\int_{R_{ij}} \text{curl}(F)\ dA = \int_{\partial R_{ij}} F \cdot\ dr$

This in turn, we can do by an essentially straightforward calculation.




We can parametrize $\partial R_{ij}$ as four pieces:

$C_1 : r_1(t) = (x_0 + t, y_0)$ , $0 \le t \le h$ ,
$C_2 : r_2(t) = (x_0 + h, y_0 + t)$ , $0 \le t \le k$ ,
$C_3 : r_3(t) = (x_0 + h - t, y_0 + k)$ , $0 \le t \le h$ ,
$C_4 : r_4(t) = (x_0, y_0 + k - t)$ , $0 \le t \le h$ , and then

$$\int_{\partial R_{ij}} F \cdot\ dr = \int_{C_1} F \cdot\ dr + \int_{C_2} F \cdot\ dr + \int_{C_3} F \cdot\ dr + \int_{C_4} F \cdot\ dr$$

But, since $r_1'(t) = \langle 1, 0\rangle$, we have

$$\int_{C_1} F\cdot\ dr$$
$$= \int_0^h F(r_1(t))\cdot\langle 1,0\rangle\ dt$$
$$= \int_0^h F_1(x_0+t, y_0)\ dt$$
$$= \int_{x_0}^{x_0+h} F_1(u, y_0)\ du$$

(using the $u$-substitution $u = x_0 + t$),
and since $r_3'(t) = \langle -1, 0\rangle$, we have

$$\int_{C_3} F\cdot\ dr$$
$$= \int_0^h F(r_3(t))\cdot\langle -1,0\rangle\ dt$$
$$= -\int_0^h F_1(x_0+h-t, y_0+k)\ dt$$
$$= \int_{x_0+h}^{x_0} F_1(u, y_0+k)\ du$$
$$= -\int_{x_0}^{x_0+h} F_1(u, y_0+k)\ du$$

(using the $u$-substitution $u = x_0 + h - t$) .

On the other hand,

$\int\int_{R_{ij}} \text{curl}(F)\ dA = \int\int_{R_{ij}} (F_2)_x - (F_1)_y\ dA$, and

$$\int\int_{R_{ij}} -(F_1)_y\ dA$$
$$= -\int_{x_0}^{x_0+h}\int_{y_0}^{y_0+k} (F_1)_y\ dy\ dx$$
$$= -\int_{x_0}^{x_0+h} (F_1(x,y)|_{y_0}^{y_0+k})\ dx$$
$$= -\int_{x_0}^{x_0+h} F_1(x, y_0+k) - F_1(x, y_0)\ dx$$
$$= -\int_{x_0}^{x_0+h} F_1(u, y_0+k)\ du + \int_{x_0}^{x_0+h} F_1(u, y_0)\ du$$
$$= \int_{x_0}^{x_0+h} F_1(u, y_0)\ du - \int_{x_0}^{x_0+h} F_1(u, y_0+k)\ du$$
$$= \int_{C_1} F\cdot\ dr + \int_{C_3} F\cdot\ dr$$

An entirely similar calculation [exercise...] shows that

$\int\int_{R_{ij}} (F_2)_x\ dA = \int_{C_2} F\cdot\ dr + \int_{C_4} F\cdot\ dr$

and so:

$$\int\int_{R_{ij}} \text{curl}(F)\ dA$$
$$= \int\int_{R_{ij}} (F_2)_x - (F_1)_y\ dA$$
$$= \int\int_{R_{ij}} (F_2)_x\ dA + \int\int_{R_{ij}} -(F_1)_y\ dA$$
$$= \left(\int_{C_2} F\cdot\ dr + \int_{C_4} F\cdot\ dr\right) + \left(\int_{C_1} F\cdot\ dr + \int_{C_3} F\cdot\ dr\right)$$
$$= \int_{C_1} F\cdot\ dr + \int_{C_2} F\cdot\ dr + \int_{C_3} F\cdot\ dr + \int_{C_4} F\cdot\ dr$$
$$= \int_{\partial R_{ij}} F\cdot\ dr$$

as desired!