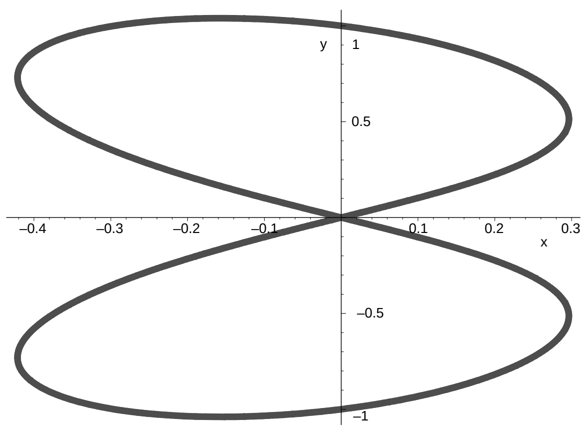
(Implicit) differentiation at a saddle point: an application of L`Hôpital's rule

We learned, for a function $y = y(x)$ defined implicitly by a function of two variables $f(x, y) = c$, that what we learn as implicit differentiation in Calculus I is in essence the multivariate Chain Rule. That is, if $f(x, y) = c$ and $x = x(t) = t$ and $y = y(t)$, then as a function of t, $z = f(x(t), y(t)) = f(t, y(t)) = c$ is constant, so

$$0 = \frac{dz}{dt} = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt} = f_x + f_y\frac{dy}{dt}, \text{ so } \frac{dy}{dx} = \frac{dy}{dt}\Big|_{t=x} = \frac{-f_x}{f_y}$$

But what happens if $f_x = f_y = 0$? This must happen, for example, where a level curve $f(x, y) = c$ crosses itself (as we would have at a saddle point for the associated function $z = f(x, y)$), since then there are 'really' two tangent slopes, and a single number cannot give both answers!



The answer is that, if $\frac{dy}{dx}$ is continuous (as most level curves we would draw do suggest), and f_x and f_y are differentiable, then L'Hopital's Rule can be applied. In one of the rule's most basic forms, we then have

$$\frac{dy}{dx}\Big|_{x=x_0} = \lim_{x\to x_0}\frac{-f_x}{f_y} = \frac{\frac{d}{dx}\left(-\frac{\partial f}{\partial x}\right)}{\frac{d}{dx}\left(\frac{\partial f}{\partial y}\right)}\Big|_{x=x_0}$$

But these are quantities that we can compute using the Chain Rule again!

$$\frac{d}{dx}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right) + \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right)\frac{dy}{dx} = f_{xx} + f_{xy}\frac{dy}{dx}$$

and

$$\frac{d}{dx}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) + \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right)\frac{dy}{dx} = f_{yx} + f_{yy}\frac{dy}{dx}$$

Setting $m = \frac{dy}{dx}$, for brevity, this yields

$$m = -\frac{f_{xx} + f_{xy}m}{f_{yx} + f_{yy}m}, \qquad \text{so} \qquad f_{yy}m^2 + (f_{xy} + f_{yx})m + f_{xx} = 0$$

Since $f_{xy} = f_{yx}$, this becomes $f_{yy}m^2 + 2f_{xy}m + f_{xx} = 0$, which is a quadratic equation in m yielding <u>two</u> solutions (just as we need!)

$$m = \frac{-2f_{xy} \pm \sqrt{(2f_{xy})^2 - 4f_{xx}f_{yy}}}{2f_{yy}} = \frac{-f_{xy} \pm \sqrt{(f_{xy})^2 - f_{xx}f_{yy}}}{f_{yy}}$$

Notice that since a level curve that crosses itself represents a saddle point of the function $z = f(x, y)$, the Hessian $H = f_{xx}f_{yy} - (f_{xy})^2$ <u>should</u> be negative at the crossing, and so the quantity inside of the square root is, in fact, positive!

We illustrate this with an example:

For the function $f(x, y) = (x^2 + y^2)^2 + (1 - x)(x^2 - y^2)$, the graph of $f(x, y) = 0$ has a double point at $(0, 0)$ (see the figure above). Since

$$f_x = 2(x^2 + y^2)2x - (x^2 - y^2) + (1 - x)(2x) \text{ and}$$
$$f_y = 2(x^2 + y^2)2y + (1 - x)(-2y)$$

are both 0 at (0.0), we can use the above approach to compute the two values of dy/dx. A routine computation finds that

$$f_{xx} = 12x^2 + 4y^2 - 6x + 2$$
$$f_{xy} = 8xy + 2y$$
$$f_{yy} = 12y^2 + 4x^2 + 2x - 2$$

So at $(x, y) = (0, 0$ we have two slopes m, which are the solutions to $-2m^2 + 2 = 0$; that is, $m = 1$ and $m = -1$.