Math 208H, Section 1

Practice problems for Exam 1 (Solutions)

1. Find the **sine** of the angle between the vectors (1,−1,2) and (1,2,1) .

We can use the dot product (dividing by lengths) to compute the cosine of the angle, and then from that the sine. Or we can use $|\vec{u}\times\vec{v}| = |\vec{u}||\vec{v}|\sin(\theta)$ to compute the sine, by finding the cross product and computing lengths.

$\sin(\theta) = \sqrt{(-5)^2+1^2+3^2}/(\sqrt{1^2+(-1)^2+2^2}\cdot\sqrt{1^2+2^2+1^2}) = \sqrt{35}/(\sqrt{6}\cdot\sqrt{6}) = \sqrt{35}/6$

This is consistent with $\cos(\theta) = (1\cdot 1+(-1)\cdot 2+2\cdot 1)/(\sqrt{6}\cdot\sqrt{6}) = 1/6$.

2. Find a vector of length 3 that is perpendicular to both

$$\vec{v} = \langle 1,3,5\rangle \text{ and } \vec{w} = \langle 2,1,-1\rangle \ .$$

A vector perpendicular to both is given by the cross product, so we compute

$$\vec{v}\times\vec{w} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 3 & 5 \\ 2 & 1 & -1 \end{vmatrix} = \begin{vmatrix} 3 & 5 \\ 1 & -1 \end{vmatrix}\vec{i} - \begin{vmatrix} 1 & 5 \\ 2 & -1 \end{vmatrix}\vec{j} + \begin{vmatrix} 1 & 3 \\ 2 & 1 \end{vmatrix}\vec{k}$$

$$= \langle -3-5, -(-1-10), 1-6\rangle = \langle -8,11,-5\rangle$$

[We can test that this is perpendicular to the two vectors by computing dot products...]

This vector has length $\sqrt{64+121+25} = \sqrt{210}$; since we want a vector of length 3, we take the appropriate scalar multiple:

$\vec{N} = \dfrac{3}{\sqrt{210}}\langle -8,11,-5\rangle$ has length 3 and is $\perp$ tp $\vec{v}$ and $\vec{w}$. [Its negative also works...]

3. Show that if the vectors $\vec{\mathbf{v}}=(a_1,a_2,a_3)$ and $\vec{\mathbf{w}}=(b_1,b_2,b_3)$ have the same length, then the vectors $\vec{\mathbf{v}}+\vec{\mathbf{w}}$ and $\vec{\mathbf{v}}-\vec{\mathbf{w}}$ are perpendicular to one another.

We wish to know that $(\vec{\mathbf{v}}+\vec{\mathbf{w}})\circ(\vec{\mathbf{v}}-\vec{\mathbf{w}}) = 0$. But expanding this out, we find that it is equal to $\vec{\mathbf{v}}\circ\vec{\mathbf{v}}-\vec{\mathbf{w}}\circ\vec{\mathbf{w}}$. This will be equal to 0 precisely when $|\vec{v}|^2 = \vec{\mathbf{v}}\circ\vec{\mathbf{v}} = \vec{\mathbf{w}}\circ\vec{\mathbf{w}} = |\vec{w}|^2$. This in turn, means that $\vec{\mathbf{v}}$ and $\vec{\mathbf{w}}$ have the same length.

4. Find the equation of the plane in 3-space which passes through the three points $(1,2,1)$, $(6,1,2)$, and $(9,-2,1)$. Does the point $(3,2,1)$ lie on this plane?

To find the equation, we need a point and a normal vector; the normal can be found by a cross product. $\vec{N} = \vec{PQ}\times\vec{PR} = (5,-1,1)\times(8,-4,0) = (4,8,-12)$. Then the equation is $(4,8,-12)\circ(x-1,y-2,z-1) = 0$, or $4x+8y-12z = 8$ (or $x+2y-3z = 2$ (!)). [Check: the 3 points satisfy the equation!] Checking, $4\cdot 3+8\cdot 2-12\cdot 1 = 16 \neq 8$, so the point does not lie on the plane.

5. Find the partial derivatives of the following functions:

(a) $f(x, y, z) = x\tan(2x + yz)$

We have $f_x = \tan(2x + yz) + x\sec^2(2x + yz) \cdot 2$, $f_y = x\sec^2(2x + yz) \cdot z$, and $f_z = x\sec^2(2x + yz) \cdot y$.

(b) $g(x, y) = \dfrac{x^2y - ty^4}{\sin(3y) + 4}$ We have $g_x = \dfrac{(2xy)(\sin(3y) + 4) - (x^2y - ty^4)(0)}{(\sin(3y) + 4)^2}$, and $g_y = \dfrac{(x^2 - 4ty^3)(\sin(3y) + 4) - (x^2y - ty^4)(3\cos(3y))}{(\sin(3y) + 4)^2}$. Since the question didn't ask us to do anything with these, why simplify them?

6. Find the equation of the tangent plane to the graph of the equation $f(x, y, z) = xy^2 + x^2z - xyz = 5$, at the point $(-1, 1, 3)$.

$f_x = y^2 + 2xz - yz$, $f_y = 2xy + x^2 - xz$, and $f_z = x^2 - xy$. The normal vector to the plane will be $(f_x(-1, 1, 3), f_y(-1, 1, 3), f_z(-1, 1, 3)) = (1-6-3, -2+1+3, 1+1) = (-8, 2, 2)$. Together with the point of tangency, this gives us the equation
$-8(x - (-1)) + 2(y - 1) + 2(z - 3) = 0$, or $-8x + 2y + 2z = 16$, or $4x - y - z = -8$.

7. Calculate the first and second partial derivatives of the function $h(x, y) = \dfrac{\sin(x + y)}{y}$

It may help a bit to write this function as $h(x, y) = y^{-1}\sin(x + y)$. Then we have
$h_x = y^{-1}\cos(x + y) \cdot 1 = y^{-1}\cos(x + y)$,
$h_y = -y^{-2}\sin(x + y) + y^{-1}\cos(x + y) \cdot 1 = -y^{-2}\sin(x + y) + y^{-1}\cos(x + y)$. Then
$h_{xx} = (h_x)_x = y^{-1}(-\sin(x + y) \cdot 1) = -y^{-1}\sin(x + y)$
$h_{yx} = h_{xy} = (h_x)_y = -y^{-2}\cos(x + y) + y^{-1}(-\sin(x + y) \cdot 1)$
$= -y^{-2}\cos(x + y) - y^{-1}\sin(x + y)$
$h_{yy} = (h_y)_y$
$= [2y^{-3}\sin(x + y) - y^{-2}(\cos(x + y) \cdot 1)] + [-y^{-2}\cos(x + y) + y^{-1}(-\sin(x + y) \cdot 1)]$
Again, we don't want to do anything with it, so why bother simplifying it...

8. In which direction is the function $f(x, y) = x^4y - 3x^2y^2$ increasing the fastest, at the point (1,2) ? In which directio**n**s is the function *neither* increasing *nor* decreasing?

f increases fastest in the direction of the gradient, so we compute:
$\nabla f = (4x^3y - 6xy^2, x^4 - 6x^2y)$, which at (1,2) gives $\vec{v} = (8 - 24, 1 - 12) = (-16, -11)$. This is the drection of fastest increase (you can divide by its length if you want a unit vector...).

For no increase/decrease, what we want is $D_{\vec{w}}f = \nabla f \circ \vec{w} = 0$, so we want $(-16, 11) \circ (\alpha, \beta) = -16\alpha - 11\beta = 0$; we can do this, for example, with $\vec{w} = (\alpha, \beta) = (11, -16)$. [There are many other answers, all scalar multiples of this one.]

9. If $f(x,y) = x^2y^5 - x + 3y - 4$, $x = x(u,v) = \dfrac{u}{u+v}$ and $y = y(u,v) = uv - u$, use the Chain Rule to find $\dfrac{\partial f}{\partial u}$ when $u = 1$ and v=0 .

First, when $(u,v) = (1,0)$, then $x = 1/(1+0) = 1$ and $y = 1 \cdot 0 - 1 = -1$. From the chain rule, we know that $f_u = f_x x_u + f_y y_u$, evaluated at $(x,y) = (1,-1)$ and $(u,v) = (1,0)$. We compute:

$f_x = 2xy^5 - 1 = -2-1 = -3$, $f_y = 5x^2y^4 + 3 = 5+3 = 8$, $x_u = \dfrac{(1)(u+v) - (u)(1)}{(u+v)^2} = \dfrac{v}{(u+v)^2} = 0$, and $y_u = v - 1 = 0 - 1 = -1$; so at $(u,v) = (1,0)$ we have $f_u(1,0) = (-3)(0) + (8)(-1) = -8$.

10. If $f(x,y) = \dfrac{x^2y}{x+y}$, and $\gamma(t) = (x(t), y(t))$ is a parametrized curve in the domain of f with $\gamma(0) = (2,-1)$ and $\gamma'(0) = (3,5)$, then what is $\dfrac{d}{dt} f(\gamma(t))\Big|_{t=0}$?

By the chain rule, $\dfrac{df}{dt} = f_x x_t + f_y y_t$. We compute: $f_x = \dfrac{(2xy)(x+y) - (x^2y)(1)}{(x+y)^2}$ and $f_y = \dfrac{(x^2)(x+y) - (x^2y)(1)}{(x+y)^2}$.

At $(2,-1)$, these are $f_x = \dfrac{(-4)(1) - (-4)(1)}{(1)^2} = 0$ and $f_y = \dfrac{(4)(1) - (-4)(1)}{(1)^2} = 8$, so

$\dfrac{df}{dt} = f_x x_t + f_y y_t = (0)(3) + (8)(5) = 40$.

11. Find the **second** partial derivatives of the function $h(x,y) = x\sin(xy^2)$.

We compute: $h_x = (1)(\sin(xy^2)) + (x)(\cos(xy^2))(y^2) = \sin(xy^2) + xy^2\cos(xy^2)$
$h_y = x(\cos(xy^2))(2xy) = 2x^2y\cos(xy^2)$. Then for the second partials:

$h_{xx} - (h_x)_x = (\cos(xy^2))(y^2) + [(y^2)(\cos(xy^2)) + (xy^2)(-\sin(xy^2))(y^2)]$
$= 2y^2\cos(xy^2) - xy^4\sin(xy^2)$

$h_{xy} = h_{yx} = (h_y)_x = (4xy)(\cos(xy^2)) + (2x^2y)(-\sin(xy^2))(y^2)$
$= 4xy\cos(xy^2) - 2x^2y^3\sin(xy^2)$

$h_{yy} = (h_y)_y = (2x^2)(\cos(xy^2)) + (2x^2y)(-\sin(xy^2))(2xy)$
$= 2x^2\cos(xy^2) - 4x^3y^2\sin(xy^2)$