Solutions

Name:

Math 208H, Section 2

Final Exam

Show all work. How you get your answer is just as important, if not more important, than the answer itself. If you think it, write it!

1. (15 pts.) Find the length of the parametrized curve

$$\vec{r}(t) = (t^6 \cos t, t^6 \sin t) \quad , \quad 0 \le t \le \pi$$

$$r'(t) = (6t^5\cos t - t^6\sin t,\ 6t^5\sin t + t^6\cos t)$$

$$\text{Length} = \int_0^\pi \left((6t^5\cos t - t^6\sin t)^2 + (6t^5\sin t + t^6\cos t)^2 \right)^{1/2} dt$$

$$= \int_0^\pi \left(36t^{10}\cos^2 t - \cancel{12t^{11}\cos t\sin t} + t^{12}\sin^2 t + 36t^{10}\sin^2 t + \cancel{12t^{11}\sin t\cos t} + t^{12}\cos^2 t \right)^{1/2} dt$$

$$= \int_0^\pi \left(36t^{10}(\cos^2 t + \sin^2 t) + t^{12}(\sin^2 t + \cos^2 t) \right)^{1/2} dt$$

$$= \int_0^\pi (36t^{10} + t^{12})^{1/2} dt = \int_0^\pi (t^{10})^{1/2}(36 + t^2)^{1/2} dt = \int_0^\pi t^5 (t^2+36)^{1/2} dt$$

$$= \frac{1}{2}\int_0^\pi (t^2)^2 (t^2+36)^{1/2}(2t)\,dt$$

$u = t^2 + 36$ ~~u=0~~ $t=0,\ u=36$

$du = 2t\,dt$ $\quad t=\pi,\ u=\pi^2+36$

$t^2 = u - 36$

$$\to\ = \frac{1}{2}\int_{36}^{\pi^2+36} (u-36)^2 u^{1/2}\,du$$

$$= \frac{1}{2}\int_{36}^{\pi^2+36} u^{5/2} - 72u^{3/2} + 36^2 u^{1/2}\,du = \frac{1}{2}\left(\frac{2}{7}u^{7/2} - 72\cdot\frac{2}{5}u^{5/2} + 36^2\cdot\frac{2}{3}u^{3/2} \right)\Big|_{36}^{\pi^2+36}$$

$$= \frac{1}{2}\Big[\left(\frac{2}{7}(\pi^2+36)^{7/2} - 72\cdot\frac{2}{5}(\pi^2+36)^{5/2} + 36^2\cdot\frac{2}{3}(\pi^2+36)^{3/2} \right) - \left(\frac{2}{7}(36)^{7/2} - 72\cdot\frac{2}{5}(36)^{5/2} + 36^2\cdot\frac{2}{3}(36)^{3/2} \right) \Big]$$

2. (15 pts.) Find the equation of the plane tangent to the graph of

$$z = f(x, y) = xe^y - \cos(2x + y)$$

at $(0, 0, -1)$

In what direction is this plane tilting up the most?

$$f_x = e^y + 2\sin(2x+y)$$

$$f_y = xe^y + \sin(2x+y)$$

At $(0,0)$:

$$f_x = e^0 + 2\sin(0) = 1$$

$$f_y = 0 \cdot e^0 + \sin(0) = 0 + 0 = 0$$

$$z - (-1) = 1 \cdot (x - 0) + 0 \cdot (y - 0) = x$$

$$\boxed{z = x - 1}$$

Most tilting? Fastest increase!

$$= \nabla f(0,0) = (f_x(0,0), f_y(0,0)) = \underline{\underline{(1,0)}}$$

3. (20 pts.) Find the critical points of the function

$$z = g(x, y) = x^2y^3 - 3y - 2x$$

and for each, determine if it is a local max, local min, or saddle point.

$g_x = 2xy^3 - 0 - 2 = 0$, $2xy^3 = 2$, $xy^3 = 1$

$g_y = 3x^2y^2 - 3 - 0 = 0$, $3x^2y^2 = 3$ $x^2y^2 = 1$

$x = x \cdot 1 = x(xy^3) = x^2y^3 = (x^2y^2)y = 1 \cdot y = y$

$\underline{y = x}$

$x \cdot x^3 = 1$, $x^4 = 1$, $x = 1, -1$

$(1,1)$, $(-1,-1)$ critical points

$\cancel{f_{xx}}$ $g_{xx} = 2y^3$, $g_{yy} = 6x^2y$, $g_{xy} = 6xy^2$

$D = g_{xx}g_{yy} - (g_{xy})^2 = (2y^3)(6x^2y) - (6xy^2)^2$
$= 12x^2y^4 - 36x^2y^4 = -24x^2y^4$

At $(1,1)$: $D = -24 \cdot 1^2 \cdot 1^4 = -24 < 0$ $\underline{\text{saddle}}$

$(-1,-1)$: $D = -24(-1)^2(-1)^4 = -24 < 0$ $\underline{\text{saddle}}$

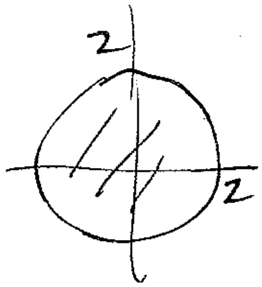
4. (15 pts.) Find the integral of the function

$$z = h(x, y) = \ln(x^2 + y^2 + 1)$$

over the region

$$R = \{(x, y) : x^2 + y^2 \le 4\}$$

polar! $x^2+y^2=r^2$



$$\iint_R h(x,y)\, dA = \int_0^{2\pi}\int_0^2 \ln(r^2+1)\cdot r\,dr\,d\theta$$

$$= \int_0^{2\pi}\left(\int_0^2 r\ln(r^2+1)\,dr\right)d\theta$$

$u = r^2+1$, $du = 2r\,dr$, $r\,dr = \frac{1}{2}du$
$r=0, u=1$; $r=2, u=5$

$$= \int_0^{2\pi}\left(\int_1^5 \tfrac{1}{2}\ln u\,du\right)d\theta = \frac{1}{2}\int_0^{2\pi}(u\ln u - u)\Big|_1^5\,d\theta$$

$$= \frac{1}{2}\int_0^{2\pi}((5\ln 5 - 5) - (1\ln 1 - 1))\,d\theta = \frac{1}{2}\int_0^{2\pi} 5\ln 5 - 4\,d\theta$$

$$= \frac{1}{2}(5\ln 5 - 4)\,\theta\Big|_0^{2\pi} = \boxed{\pi(5\ln 5 - 4)}$$

$v = \ln u \quad dw = du$

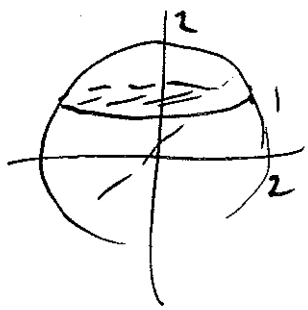
$\int \ln u\,du$; $dv = \frac{1}{u}du \quad w = u$

$= u\ln u - \int \frac{1}{u}\cdot u\,du = u\ln u - \int du = u\ln u - u$

5. (20 pts.) Find the integral of the function

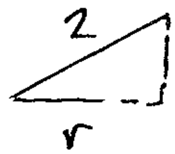
$$k(x,y,z) = z$$

over the region lying inside of the sphere of radius 2 (centered at the origin $(0,0,0)$) and above the plane $z = 1$.



cylindrical coords: $x^2+y^2+z^2 = r^2+z^2 = 4$

$z = \sqrt{4-r^2}$



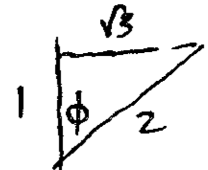
$, \; r = \sqrt{3}$.

$$\iiint_R z\,dV = \int_0^{2\pi}\int_0^{\sqrt{3}}\int_1^{\sqrt{4-r^2}} z\,dz\cdot r\,dr\,d\theta = \int_0^{2\pi}\int_0^{\sqrt{3}} r\frac{z^2}{2}\Big|_1^{\sqrt{4-r^2}} dr\,d\theta$$

$$= \int_0^{2\pi}\int_0^{\sqrt{3}} \frac{r}{2}(4-r^2) - \frac{r}{2}\,dr\,d\theta = \int_0^{2\pi}\int_0^{\sqrt{3}} \frac{3r}{2} - \frac{r^3}{2}\,dr\,d\theta$$

$$= \int_0^{2\pi} \frac{3r^2}{4} - \frac{r^4}{8}\Big|_0^{\sqrt{3}} d\theta = \int_0^{2\pi}\left(\frac{9}{4} - \frac{9}{8}\right) d\theta = \int_0^{2\pi} \frac{9}{8}\,d\theta = \boxed{\frac{9\pi}{4}}$$

or . $z = r\cos\phi = 1 \quad r = \frac{1}{\cos\phi} = \sec\phi$



$\phi = \frac{\pi}{3}$

$$\iiint_R z\,dV = \int_0^{2\pi}\int_0^{\pi/3}\int_{\sec\phi}^{2} (\rho\cos\phi)(\rho^2\sin\phi)\,d\rho\,d\phi\,d\theta$$

$$= \int_0^{2\pi}\int_0^{\pi/3} \sin\phi\cos\phi\,\frac{\rho^4}{4}\Big|_{\sec\phi}^{2} d\phi\,d\theta = \int_0^{2\pi}\int_0^{\pi/3} 4\sin\phi\cos\phi - \frac{1}{4}\frac{\sin\phi}{\cos^3\phi}\,d\phi\,d\theta$$

$$= \int_0^{2\pi}\left(2\sin^2\phi + \frac{1}{4}\frac{(\cos\phi)^{-2}}{-2}\right)\Big|_0^{\pi/3} d\theta = (2\pi)\left(2\left(\frac{\sqrt{3}}{2}\right)^2 - \frac{1}{8}\left(\frac{1}{2}\right)^{-2} - \left(0 - \frac{1}{8}(1)^{-2}\right)\right)$$

$$= 2\pi\left(\frac{3}{2} - \frac{1}{2} + \frac{1}{8}\right) = 2\pi\left(\frac{9}{8}\right) = \boxed{\frac{9\pi}{4}}$$

6. (20 pts.) Show that the vector field $\vec{F} = \langle y^2, 2xy - 1 \rangle$ is conservative, find a potential function $z = f(x, y)$ for $\vec{F}$, and use this potential function to (quickly!) find the integral of $\vec{F}$ along the path

$$\vec{r}(t) = (t\sin(2\pi t) - e^t, \ln(t^2 + 1) - 5t^2) \quad , \quad 0 \le t \le 1$$

$F_1 = y^2 \quad F_2 = 2xy - 1$ so $\text{curl}(\vec{F}) = 0$ so conservative.

$(F_2)_x = 2y = (F_1)_y$

$f(x,y) = \int y^2\,dx = xy^2 + g(y)$

$F_2 = 2xy - 1 = f_y = 2xy + g'(y) \quad g'(y) = 1 \quad g(y) = y$

$f(x,y) = \boxed{xy^2 - y = \text{potential function}}$

$\int_C \vec{F} \cdot d\vec{r} = \int_C \nabla f \cdot d\vec{r} = f(r(1)) - f(r(0))$

$r(1) = (1 \cdot \sin(2\pi) - e^1, \ln(1+1) - 5 \cdot 1^2) = (-e, \ln 2 - 5)$

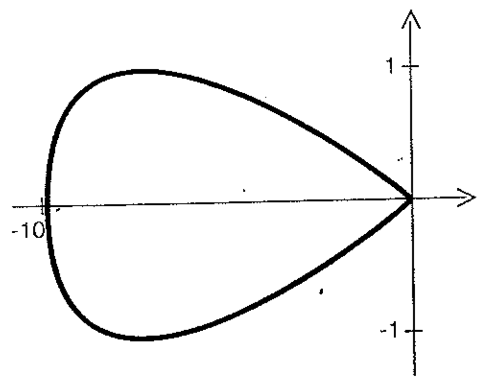
$r(0) = (0 \cdot \sin(0) - e^0, \ln(0+1) - 5 \cdot 0^2) = (-1, 0)$

$\int_C \vec{F} \cdot d\vec{r} = ((-e)(\ln 2 - 5)^2 - (\ln 2 - 5)) - ((-1)(0)^2 - 0)$

$= \boxed{-e(\ln 2 - 5)^2 - (\ln 2 - 5)}$

7. (15 pts.) Use Green's Theorem to find the area of the region enclosed by the curve

$$\vec{r}(t) = (t^2 - 2\pi t, \sin t) \quad , \quad 0 \le t \le 2\pi$$



$$\text{Area} = \iint_R 1\,dA = \int_C \langle -y, 0\rangle \cdot d\vec{r}$$

$$r'(t) = (2t - 2\pi, \cos t)$$

$$\text{Area} = \int_0^{2\pi} (-\sin t, 0) \cdot (2t - 2\pi, \cos t)\, dt$$

$$= \int_0^{2\pi} 2\pi \sin t - 2t \sin t \, dt$$

$$=$$

$$\int t \sin t\, dt \begin{cases} u = t & dv = \sin t\, dt \\ du = dt & v = -\cos t \end{cases}$$

$$= -t\cos t + \int \cos t\, dt = -t \cos t + \sin t$$

$$\text{Area} = -2\pi \cos t - 2(-t\cos t + \sin t)\Big|_0^{2\pi}$$

$$= 2t\cos t - 2\pi \cos t \; 2\sin t \Big|_0^{2\pi}$$

$$= (2\cdot 2\pi \cdot 1 - 2\pi \cdot 1 - 0) - (0 \cdot 1 - 2\pi \cdot 1 - 0)$$

$$= \boxed{\cancel{2\pi}\; 4\pi}$$

Area $= \int_C \langle 0, x\rangle \cdot d\vec{r}$ gives the same answer.... but takes a little longer.

8. (20 pts.) Find the flux of the vector field

$$\vec{G} = \langle x^2, xz, y \rangle$$

through that part of the graph of

$$z = f(x, y) = xy$$

lying over the rectangle

$$1 \le x \le 3 \quad , \quad 0 \le y \le 3$$

$f_x = y$

$f_y = x$

$$\iint_{\Sigma} \vec{G} \cdot \vec{N}\, dA = \int_1^3 \int_0^3 \langle x^2, x(xy), y \rangle \cdot \langle -y, -x, 1 \rangle \, dy\, dx$$

$$= \int_1^3 \int_0^3 -x^2y - x^3y + y \, dy\, dx$$

$$= \int_1^3 -\frac{x^2y^2}{2} - \frac{x^3y^2}{2} + \frac{y^2}{2} \Big|_0^3 dx$$

$$= \int_1^3 \left(-\frac{9}{2}x^2 - \frac{9}{2}x^3 + \frac{9}{2}\right) - (0 - 0 - 0)\, dx$$

$$= \int_1^3 \frac{9}{2} - \frac{9}{2}x^2 - \frac{9}{2}x^3 \, dx = \frac{9}{2}x - \frac{9}{6}x^3 - \frac{9}{8}x^4 \Big|_1^3$$

$$\boxed{= \left(\frac{9}{2}(3) - \frac{9}{6}(3)^3 - \frac{9}{8}(3)^4\right) - \left(\frac{9}{2} - \frac{9}{6} - \frac{9}{8}\right)} = -120$$

Solutions ✓

Name:

Math 208H, Section 1

Final Exam

Show all work. How you get your answer is just as important, if not more important, than the answer itself. If you think it, write it!

A complete exam consists of solutions to the first eight (8) problems, **together with solutions to three (3) of the last four (4) problems** (numbers 9 through 12).

1. (10 pts.) Find the orthogonal projection of the vector $\vec{v}$ = (3,1,2) onto the vector $\vec{w}$ = (-1,4,2) .

$$\mathrm{proj}_{\vec{w}}(\vec{v}) = \frac{\vec{v}\cdot\vec{w}}{\vec{w}\cdot\vec{w}}\vec{w}$$

$\vec{v} = (3,1,2)$

$\vec{v}\cdot\vec{w} = -3+4+4 = 5$

$\vec{w}\cdot\vec{w} = 1+16+4 = 21$

$$\mathrm{proj}_{\vec{w}}(\vec{v}) = \frac{5}{21}(-1,4,2) = \boxed{\left(\frac{-5}{21}, \frac{20}{21}, \frac{10}{21}\right)}$$

Check: $\vec{v} - \mathrm{proj}_{\vec{w}}(\vec{v}) = \left(\frac{68}{21}, \frac{1}{21}, \frac{32}{21}\right)$

$\frac{1}{21}(-68+4+64) = 0$ ✓

2. (10 pts.) Find the equation of the plane passing through the points

(1,1,1), (2,1,3), and (-1,2,1)

P Q R

$\vec{PQ} = (1,0,2)$

$\vec{PR} = (-2,1,0)$

$\vec{n} = \vec{PQ} \times \vec{PR} = \begin{vmatrix} i & j & k \\ 1 & 0 & 2 \\ -2 & 1 & 0 \end{vmatrix} = (-2,-4,1)$

Check $-2+0+2=0$ ✓
$4-4+0=0$ ✓

eqn:

$(-2,-4,1)\cdot(x-1,y-1,z-1)=0$

$-2(x-1)-4(y-1)+1(z-1)=0$

$z = 1 + 2(x-1) + 4(y-1)$

$\boxed{z = 2x+4y-5}$

Check: $1 = 2+4-5$ ✓
$3 = 4+4-5$ ✓
$1 = -2+8-5$ ✓

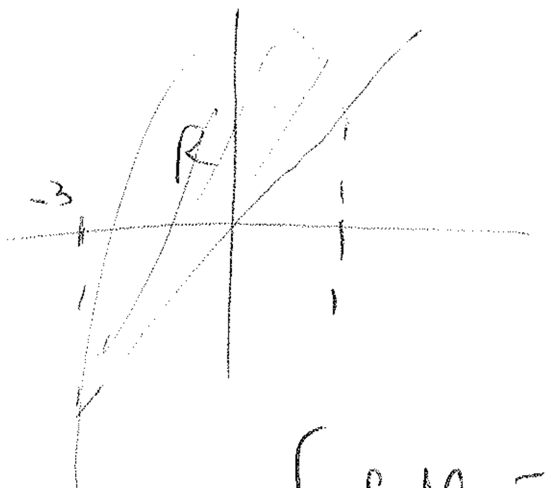
X

4. (10 pts.) Find the integral of the function

$$f(x, y) = xy^2$$

over the region in the plane lying between the graphs of

$$a(x) = 2x \qquad \text{and} \qquad b(x) = 3 - x^2$$



$2x = 3 - x^2 \qquad x^2 + 2x - 3 = 0$

$(x+3)(x-1) = 0$

$x = -3, \; x = 1$

$$\int_R f \, dA = \int_{-3}^{1} \int_{2x}^{3-x^2} xy^2 \, dy \, dx$$

$8x^4$

$$= \int_{-3}^{1} \frac{x}{3} y^3 \Big|_{y=2x}^{y=3-x^2} dx = \frac{1}{3} \int_{-3}^{1} x(3-x^2)^3 - x(2x)^3 \, dx$$

$$= \frac{1}{3} \left(-\frac{1}{2} \frac{(3-x^2)^4}{4} - \frac{8}{5} x^5 \right) \Big|_{-3}^{1}$$

$$= \frac{1}{3} \left[\left(-\frac{1}{8} 2^4 - \frac{8}{5} 1^5 \right) - \left(-\frac{1}{8} (-6)^4 - \frac{8}{5} (-3)^5 \right) \right]$$

5. (10 pts.) Find the integral of the vector field

$$F(x,y) = (xy, x+y)$$

along the parametrized curve

$$\vec{r}(t) = (e^t, e^{2t}) \qquad 0 \le t \le 1)$$

$\vec{r}\,'(t) = (e^t, 2e^{2t})$

$\vec{F}(\vec{r}(t)) = (e^t e^{2t}, e^t + e^{2t})$

$\int_C F \cdot d\vec{r} = \int_0^1 e^t e^{2t} e^t + (e^t + e^{2t}) 2e^{2t} \, dt$

$= \int_0^1 e^{4t} + 2e^{3t} + 2e^{4t} \, dt$

$= \int_0^1 3e^{4t} + 2e^{3t} \, dt = \frac{3}{4}e^{4t} + \frac{2}{3}e^{3t} \Big|_0^1$

$= \left(\frac{3}{4}e^4 + \frac{2}{3}e^3\right) - \left(\frac{3}{4} + \frac{2}{3}\right)$

$\frac{17}{12}$

6. (4 pts. each) Which of the following vector fields are **gradient** vector fields?

(a) $F(x,y) = (y\sin(xy), x\sin(xy))$

$\quad\quad\quad\quad A \quad\quad\quad B$

$B_x = \sin(xy) + x\cos(xy)\,y$

$A_y = \sin(xy) + y\cos(xy)\,x$

equal, so F is a gradient v.f.

$\quad\quad A \quad B \quad C$

(b) $G(x,y,z) = (x^2y, z^2+x, 2yz)$

$\mathrm{curl}(G) = (C_y - B_z, -(C_x - A_z), B_x - A_y)$

$= (2z - 2z, -(0-0), 1-x^2) \neq (0,0,0)$

so not a gradient v.f.

$\quad\quad A \quad\quad B \quad\quad C$

(c) $H(x,y,z) = (y+y^2z, x+2xyz, xy^2)$

$\mathrm{curl}(H) = (2xy - 2xy, -(y^2-y^2), (1+2yz) - (1-2yz))$

$= (0,0,0)$

so H is a gradient v.f.

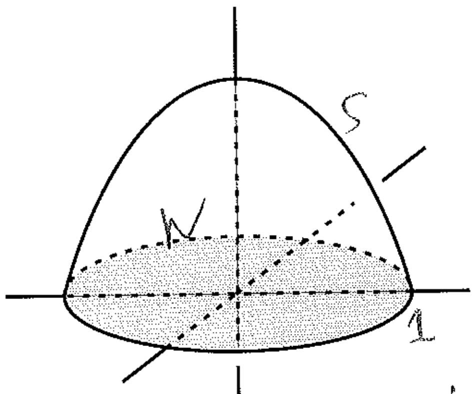
7. (15 pts.) Use the Divergence Theorem to find the flux integral of the vector field

$$F(x,y,z) = (y, xy, z)$$

through the boundary of the region lying under the graph of

$$f(x,y) = 1 - x^2 - y^2$$

and above the x-y plane (see figure).



$\int_S \vec{F}\cdot d\vec{A} = \int_W \operatorname{div}\vec{F}\, dV$

$\operatorname{div}\vec{F} = 0 + x + 1 = x + 1$

$1 - x^2 - y^2 = 1 - (x^2+y^2) = 1 - r^2$

Cylindrical coords!

$x + 1 = r\cos\theta + 1$

$0 \le z \le 1 - r^2$

$0 \le r \le 1$

$0 \le \theta \le 2\pi$

$\int_W \operatorname{div}\vec{F}\, dV = \int_0^{2\pi}\int_0^1\int_0^{1-r^2} (r\cos\theta + 1)\, r\, dz\, dr\, d\theta$

$= \int_0^{2\pi}\int_0^1 r(r\cos\theta + 1)\, z\Big|_0^{1-r^2} = \int_0^{2\pi}\int_0^1 (r^2\cos\theta + r)(1 - r^2)\, dr\, d\theta$

$= \int_0^{2\pi}\int_0^1 \cos\theta(r^2 - r^4) + r - r^3\, dr\, d\theta = \int_0^{2\pi} \cos\theta\left(\frac{r^3}{3} - \frac{r^5}{5}\right)\Big|_0^1 + \frac{r^2}{2} - \frac{r^4}{4}\Big|_0^1 d\theta$

$= \int_0^{2\pi} \frac{2}{15}\cos\theta + \frac{1}{4}\, d\theta = \frac{2}{15}\sin\theta + \frac{1}{4}\theta\Big|_0^{2\pi} = (0 + \cancel{2\pi}) - (0 + 0)$ $\pi/2$

$= \boxed{\cancel{2\pi}}$ $\pi/2$

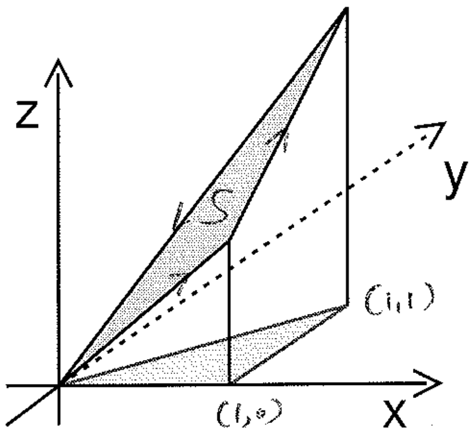
8. (13 pts.) Use Stokes Theorem to find the line integral of the vector field

$$F(x,y,z) = (xy, xz, yz)$$

around the triangle with vertices

$$(0,0,0),\ (1,0,1),\ \text{and } (1,1,2)$$

(see figure).



$$\int_C F \cdot d\vec{r} = \int_S \operatorname{curl}\vec{F} \cdot d\vec{A}$$

$$\operatorname{curl}\vec{F} = (z-x, -(0-0), z-x) = (z-x, 0, z-x)$$

S: $z = ax+by+c$

$0 = 0+0+c \qquad c=0$

$1 = a+0+0 \qquad a=1$

$2 = 1+b+0 \qquad b=1$

$\boxed{z = x+y} = f(x,y)$

$\vec{n}\,dA = (-1,-1,1)\,dx\,dy$

$(\operatorname{curl}\vec{F}) \cdot \vec{n}\,dA = -(z-x) - (0) + (z-x) = 0 \quad (!)$

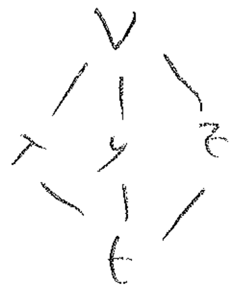
$$\int_S \operatorname{curl}\vec{F} \cdot d\vec{A} = \int_0^1\int_0^x 0\,dy\,dx = \boxed{\underline{\underline{0}}}\ !$$

9. (15 pts.) Imagine a box with side lengths $x = 2$, $y = 3$, and $z = 4$, and these lengths all change with time. How fast is the volume of the box changing, if

$$\frac{dx}{dt} = -3, \quad \frac{dy}{dt} = -2, \text{ and } \frac{dz}{dt} = -1 \ ?$$

+3

$V = xyz$



$V_x = yz$

$V_y = xz$

$V_z = xy$

$\frac{dV}{dt} = V_x x_t + V_y y_t + V_z z_t$

$x=2, y=3, z=4 \qquad V_x = 12, \ V_y = 8, \ V_z = 6$

$x_t = +3 \quad y_t = -2, \ z_t = -1$

$\frac{dV}{dt} = (12)(3) + (8)(-2) + 6(-1)$

$= 36 - 16 - 6 = \boxed{14}$

10. (15 pts.) Find the critical points of the function

$$f(x,y) = x^3y^2 - 6x^2 - y^2$$

and for each, determine if it is a rel max, rel min, or saddle point. Does the function have a global maximum?

$f_x = 3x^2y^2 - 12x = 0$

$f_y = 2x^3y - 2y = 0$

$3x(xy^2 - 4) = 0$

$2y(x^3 - 1) = 0 \longrightarrow x^3 = 1$ or $y = 0$

$x^3 = 1 \to x = 1 \to 3(y^2 - 4) = 0 \to y^2 = 4 \longrightarrow y = \pm 2$

$(1,-2), (1,2)$

$y = 0 \to -12x = 0 \to x = 0$

$(0,0)$

$f_{xx} = 6xy^2 - 12$

$f_{xy} = 6x^2y$

$f_{yy} = 2x^3 - 2$

	$(1,-2)$	$(1,2)$	$(0,0)$
f_{xx}	12	12	-12
f_{yy}	0	0	-2
f_{xy}	-12	-12	0
D	$-144 < 0$	$-144 < 0$	$24 > 0$, $f_{xx} < 0$
	saddle	saddle	rel max

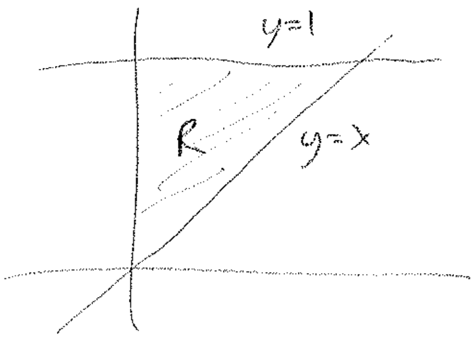
No global max!

$y = 1 \quad x \to \infty$

blows up.

11. (15 pts.) By switching the order of integration, find the integral

$$\int_0^1 \int_x^1 x e^{\frac{x^2}{y}}\, dy\, dx$$



$$\int_R f\, dA = \int_0^1 \int_0^y x\, e^{\frac{x^2}{y}}\, dx\, dy$$

$$= \int_0^1 \frac{1}{2} y\, e^{\frac{x^2}{y}} \Big|_0^y dy = \frac{1}{2}\int_0^1 y e^y - y\, dy$$

$$= \frac{1}{2}\left[\left(y e^y - e^y\right) - \frac{y^2}{2}\right]\Big|_0^1 = \frac{1}{2}\left[\left(e - e - \frac{1}{2}\right) - \left(0 - 1 - 0\right)\right]$$

$$= \frac{1}{2}\left(\frac{1}{2}\right) = \boxed{\frac{1}{4}}$$

41

Solutions

Name:

Math 208H, Section 1 Final Exam

Show all work. How you get your answer is just as important, if not more important, than the answer itself. If you think it, write it!

This exam consists of nine (9) questions on nine (9) pages. Your score on the exam will be computed from the first seven questions *plus* the better of the two scores on the final two problems. [You may work all nine if you wish, but this is not required.] All problems have equal weight.

1. Find the equation of the plane tangent to the graph of the function

$$f(x,y) = \sqrt{2x^2+y} = (2x^2+y)^{\frac{1}{2}}$$

at the point $(2,1,3)$.

$f_x = \frac{1}{2}(2x^2+y)^{-\frac{1}{2}}(4x)$

$f_y = \frac{1}{2}(2x^2+y)^{-\frac{1}{2}}(1)$

At $(2,1)$, $f_x = \frac{1}{2}(2\cdot 4+1)^{-\frac{1}{2}}(8) = 4(9)^{-\frac{1}{2}} = 4/3$

$f_y = \frac{1}{2}(2\cdot 4+1)^{-\frac{1}{2}}(1) = \frac{1}{2}\left(\frac{1}{3}\right) = 1/6$

So: tangent plane is $\boxed{z = 3 + \frac{4}{3}(x-2) + \frac{1}{6}(y-1)}$

or: normal vector is $(-f_x, -f_y, 1) = \left(-\frac{4}{3}, -\frac{1}{6}, 1\right)$

so $\left((x-2),(y-1),(z-3)\right)\cdot\left(-\frac{4}{3}, -\frac{1}{6}, 1\right) = 0$

$\boxed{-\frac{4}{3}(x-2) - \frac{1}{6}(y-1) + (z-3) = 0}$

43

2. If the temperature in a room is given by the function

$$H(x,y,z)=\frac{xy+z}{x+y},$$

use the Chain Rule to compute the *rate of change* of the temperature, as you travel along the curve $\gamma(t)=(x(t),y(t),z(t))=(t^2,2t,t^3)$, at time $t=1$.

$$H_x=\frac{(x+y)(y)-(xy+z)(1)}{(x+y)^2}$$

$$\gamma(1)=(+1^2,2,1^3)=(1,2,1)$$

$$H_y=\frac{(x+y)(x)-(xy+z)(1)}{(x+y)^2}$$

$$H_z=\frac{(x+y)(1)-(xy+z)(0)}{(x+y)^2}$$

At $(1,2,1)$, $H_x=\frac{(3)(2)-(3)(1)}{(3)^2}=\frac{6-3}{9}=\frac{1}{3}$

$$H_y=\frac{(3)(1)-(3)(1)}{3^2}=\frac{3-3}{9}=0$$

$$H_z=\frac{(3)(1)-(3)(0)}{3^2}=\frac{3}{9}=\frac{1}{3}$$

$\gamma'(t)=(2t,2,3t^2)$

$\gamma'(1)=(2,2,3)=(x_t,y_t,z_t)$

Then $\frac{dH}{dt}\Big| = H_x x_t+H_y y_t+H_z z_t=\frac{1}{3}(2)+0(2)+\frac{1}{3}(3)=\frac{2}{3}+1=\boxed{\frac{5}{3}}$

46

3. Find the point(s) on the ellipse $3x^2 + y^2 = 1$ where the function $f(x, y) = x^3y$ has its smallest (i.e., most negative) value.

$g(x,y) = 3x^2+y^2 = 1$ $\quad$ $f(x,y) = x^3y$

$\nabla g = (6x, 2y)$ $\quad$ $\nabla f = (3x^2y, x^3)$

$\nabla f = \lambda \nabla g$ $\quad$ $3x^2y = \lambda 6x$ $\quad$ $3x(xy - 2\lambda) = 0$

$x^3 = \lambda 2y$

$x = 0$ or $xy = 2\lambda$

$y^2 = 3\cdot 0^2 + y^2 = 1$ $\quad$ $\lambda = \frac{1}{2}xy$

$y = \pm 1$

$x^3 = 2\lambda y = 2\frac{1}{2}xy\cdot y$

$= xy^2$

$1 = 3x^2+y^2 = 3x^2 + x^2 = 4x^2$ $\quad$ $x(x^2-y^2) = 0$

$x^2 = \frac{1}{4} \rightarrow x = \pm\frac{1}{2}$ $\quad$ $x^2 = y^2$

$x = \pm y$

So
$(0,1)$ $\quad f = 0$
$(0,-1)$ $\quad f = 0$
$(\frac{1}{2}, -\frac{1}{2}) \rightarrow$ $\quad f = -1/16$
$(\frac{1}{2}, \frac{1}{2})$ $\quad f = 1/16$
$(-\frac{1}{2}, -\frac{1}{2})$ $\quad f = 1/16$
$(-\frac{1}{2}, \frac{1}{2})$ $\quad f = -1/6$

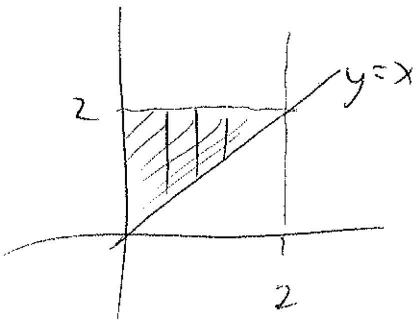
are possible

$\rightarrow$ smallest at $(\frac{1}{2}, -\frac{1}{2})$ and $(-\frac{1}{2}, \frac{1}{2})$

50

4. By *reversing* the order of integration, compute

$$\int_0^2 \int_x^2 x\sqrt{y^3+1}\, dy\, dx$$



$$= \int_0^2 \int_0^y x\,(y^3+1)^{\frac{1}{2}}\, dx\, dy$$

$$= \int_0^2 \frac{x^2}{2}(y^3+1)^{\frac{1}{2}} \Big|_0^y\, dy$$

$$= \int_0^2 \frac{y^2}{2}(y^3+1)^{\frac{1}{2}} dy = \frac{1}{6}\int_0^2 3y^2 (y^3+1)^{\frac{1}{2}} dy$$

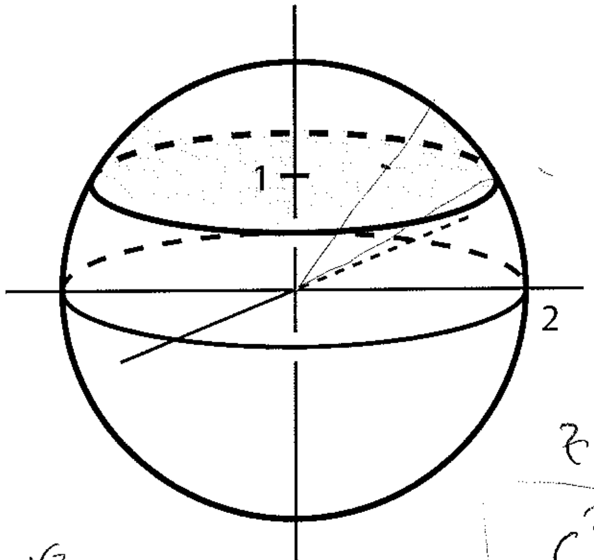
$u = y^3+1$
$du = 3y^2 dy$
$y=0 \rightarrow u=1$
$y=2 \rightarrow u = 2^3+1 = 9$

$$= \frac{1}{6}\int_1^9 u^{\frac{1}{2}} du = \frac{1}{6}\,\frac{2}{3}u^{\frac{3}{2}}\Big|_1^9$$

$$= \frac{1}{9}\left(9^{\frac{3}{2}} - 1^{\frac{3}{2}}\right) = \frac{1}{9}(3^3-1) = \frac{1}{9}(27-1) = \boxed{\frac{26}{9}}$$

S3

5. Set up **but do not compute** the triple integrals needed to find the volume of the region lying inside of the sphere $x^2 + y^2 + z^2 = 4$ and *above* the plane $z = 1$ **in both rectangular and spherical coordinates** (see figure).



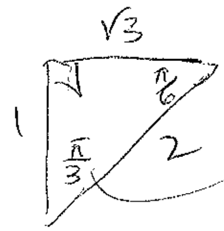
spherical:

$0 \le \theta \le 2\pi$

$0 \le \varphi \le ??$

$?? \le \rho \le 2$

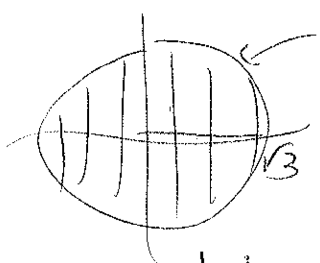
$z = \rho\cos\varphi = 1 \to \rho = \frac{1}{\cos\varphi} = \sec\varphi$



$\to \varphi \le \frac{\pi}{3}$

$$\boxed{\mathrm{Vol} = \int_0^{2\pi}\int_0^{\frac{\pi}{3}}\int_{\sec\varphi}^{2} \rho^2\sin\varphi \, d\rho\, d\varphi\, d\theta}$$

"Shadow"



$x^2+y^2=3$

$x^2+y^2+z^2=4 \quad z = \pm\sqrt{4-x^2-y^2}$

$$\boxed{\mathrm{Vol} = \int_{-\sqrt{3}}^{\sqrt{3}}\int_{-\sqrt{3-x^2}}^{\sqrt{3-x^2}}\int_{1}^{\sqrt{4-x^2-y^2}} dz\, dy\, dx.}$$

56

6. Find a potential function for the conservative vector field (in the plane)

$$\vec{F}(x,y) = (\cos x \cos y, -\sin x \sin y)$$

and use this to compute the line integral $\int_\gamma \vec{F} \circ d\vec{r}$ for the curve

$$\gamma(t) = (t\sin(\pi t), t^2\cos(\pi t)) \ , \ 0 \leq t \leq 2$$

$\gamma(2) = (2\sin(2\pi), 4\cos(2\pi))$
$= (0,4)$

$\vec{F} = \nabla f = (f_x, f_y)$ $\quad f_x = \cos x \cos y$
$\quad f_y = -\sin x \sin y$

$\gamma(0) = (0\sin(0), 0\cos(0))$
$= (0,0)$

$f(x,y) = \int \cos x \cos y \, dx = \sin x \cos y + c(y)$

$f_y = -\sin x \sin y = -\sin x \sin y + c'(y) \Longrightarrow c'(y) = 0$

take $c(y) = 0$

so $f(x,y) = \sin x \cos y$

Thus: $\int_\gamma \cancel{\vec{F}(x,y)}\, \vec{F} \cdot d\vec{r} = f(\gamma(2)) - f(\gamma(0))$

$= f(0,4) - f(0,0)$

$= \sin(0)\cos(4) - \sin(0)\cos(0)$

$= 0 \cdot \cos(4) - 0 \cdot 1 = 0 - 0 = \boxed{0}$ (!)

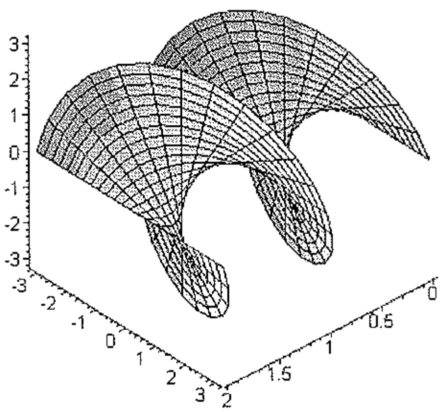
7. Set up **but do not evaluate** an iterated integral which will compute the flux integral of the vector field $\vec{F}(x, y, z) = (y, x, z)$ across the "helical spiral" Σ, parametrized by

$$T(u, v) = (u, v\cos u, v\sin u)\ ,\ \text{for } 0 \le u \le 2\pi \text{ and } -1 \le v \le 1$$

(see figure).

$$T_u = (1, -v\sin u, v\cos u)$$

$$T_v = (0, \cos u, \sin u)$$



$$T_u \times T_v = \begin{vmatrix} i & j & k \\ 1 & -v\sin u & v\cos u \\ 0 & \cos u & \sin u \end{vmatrix}$$

$$= (-v\sin^2 u - v\cos^2 u, -(\sin u - 0), (\cos u - 0))$$

$$= (-v(\sin^2 u + \cos^2 u, -\sin u, \cos u)$$

$$= (-v, -\sin u, \cos u) = \vec{N}$$

$$\vec{F}(T(u,v)) = (v\cos u, u, v\sin u)$$

$$\iint_\Sigma \vec{F} \cdot \vec{N}\, dA = \boxed{\int_0^{2\pi}\int_{-1}^{1} -v^2\cos u - u\sin u + v\sin u\cos u \, dv\, du}$$

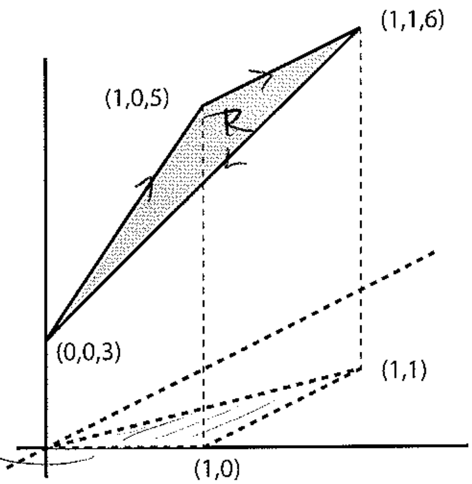
02

Note: For a complete exam it is necessary to work this problem <u>or</u> the next; both are not required. You are allowed/welcome/encouraged to work both; both will be graded.

8. Use Stokes' Theorem to compute the work done by the force field

$$\vec{G}(x,y,z) = (xy, z, xz)$$

around the edges of the triangle lying on the graph of the function $z = 2x + y + 3$, with corners at $(0,0,3)$, $(1,0,5)$, and $(1,1,6)$ (see figure).



$0 \le v \le u$
$0 \le u \le 1$

$T(u,v) = (u, v, 2u+v+3)$

$\int_\gamma \vec{G} \cdot d\vec{r} = \iint_R \operatorname{curl}\vec{G} \cdot \vec{N}\, dA$

$T_u = (1,0,2)$ $\quad T_u \times T_v = \begin{vmatrix} i & j & k \\ 1 & 0 & 2 \\ 0 & 1 & 1 \end{vmatrix}$

$T_v = (0,1,1)$

$= (-2, -(1), 1)$

$= (-2,-1,1)$

or: R is graph of $f(x,y) = 2x+y+3$

$\vec{N} = (-f_x, -f_y, 1) = (-2,-1,1)$

$\operatorname{curl}\vec{G} = \begin{vmatrix} i & j & k \\ \partial_x & \partial_y & \partial_z \\ xy & z & xz \end{vmatrix} = \left((xz)_y - (z)_z, -\left((xz)_x - (xy)_z\right), (z)_x - (xy)_y\right)$

$= (-1, -z, -x)$

$\operatorname{curl}\vec{G}(T(u,v)) = (-1, -2u-v-3, -u)$

$\iint_R \operatorname{curl}\vec{G} \cdot \vec{N}\, dA = \int_0^1\int_0^u (-1)(-2) + (-1)(-2u-v-3) + (1)(-u)\, dv\, du$

$= \int_0^1\int_0^u 2 + 2u + v + 3 - u \, dv\, du = \int_0^1\int_0^u 5 + u + v\, dv\, du$

$= \int_0^1 5v + uv + \frac{v^2}{2}\Big|_0^u du = \int_0^1 5u + u^2 + \frac{u^2}{2}\, du = \int_0^1 5u + \frac{3}{2}u^2\, du$

$= \frac{5}{2}u^2 + \frac{1}{2}u^3\Big|_0^1 = \frac{5}{2} + \frac{1}{2} = \boxed{3}$.

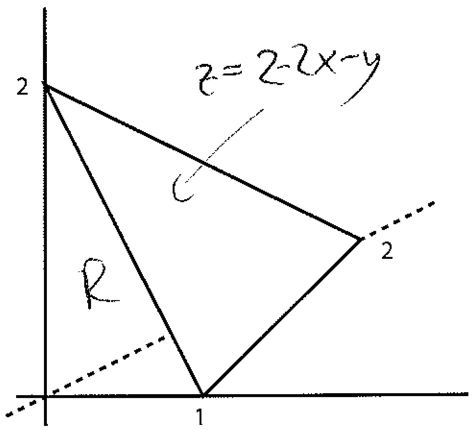
08

Note: For a complete exam it is necessary to work this problem <u>*or*</u> *the previous one; both are not required. You are allowed/welcome/encouraged to work both; both will be graded.*

9. Use the Divergence Theorem to compute the flux of the vector field

$$\vec{F}(x,y,z) = (yz, x, xz)$$

through (<u>all</u> of) the sides of the "pyramid" obtained by slicing a corner off of the first octant ($x \geq 0$, $y \geq 0$, $z \geq 0$) by the plane $2x + y + z = 2$ (see figure).



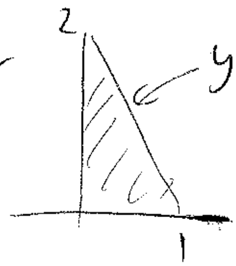
$$\operatorname{div}\vec{F} = (yz)_x + (x)_y + (xz)_z = 0 + 0 + x = x$$

$$\iint_\Sigma \vec{F}\cdot\vec{N}\,dA = \iiint_R \operatorname{div}\vec{F}\,dV$$

$$= \int_0^1\int_0^{2-2x}\int_0^{2-2x-y} x\,dz\,dy\,dx$$

R: $0 \le z \le 2-2x-y$

for



$0 \le y \le 2-2x$

$0 \le x \le 1$

$$= \int_0^1\int_0^{2-2x} xz\Big|_0^{2-2x-y} dy\,dx = \int_0^1\int_0^{2-2x} 2x - 2x^2 - xy\,dy\,dx$$

$$= \int_0^1 2xy - 2x^2y - x\frac{y^2}{2}\Big|_0^{2-2x} dx = \int_0^1 2x(2-2x) - 2x^2(2-2x) - \frac{x}{2}(2-2x)^2\,dx$$

$$= \int_0^1 (2-2x)(2x - 2x^2 - x(1-x))\,dx = \int_0^1 (2-2x)(x-x^2)\,dx$$

$$= \int_0^1 2x - 2x^2 - 2x^2 + \cancel{4}2x^3\,dx = \int_0^1 2x - 4x^2 + \cancel{4}2x^3\,dx$$

$$= x^2 - \frac{4}{3}x^3 + \frac{x^4}{2}\Big|_0^1 = 1 - \frac{4}{3} + \frac{1}{2} = \cancel{\frac{6}{3}}\frac{3}{2} - \frac{4}{3} = \cancel{\frac{2}{3}} = \frac{9-8}{6} = \boxed{\frac{1}{6}}$$

PR 13