Solutions

Name:

Math 208H, Section 1

Final Exam

Show all work. How you get your answer is just as important, if not more important, than the answer itself. If you think it, write it! The exam totals 160 points.

1. (20 pts.) Find the equation of the plane tangent to the graph of the function

$$f(x,y) = \frac{xy}{x+2y} = xy(x+2y)^{-1}$$

at the point $(1, 2, f(1,2))$. What vector is perpendicular to this plane?

$$f(1,2) = \frac{1\cdot 2}{1+2\cdot 2} = \frac{2}{5}$$

$$f_x = (y)(x+2y)^{-1} + (xy)(-1)(x+2y)^{-2}(1)$$

$$f_x(1,2) = 2\cdot 5^{-1} + (-2)(5)^{-2} = \frac{2}{5} - \frac{2}{25} = \frac{8}{25}$$

$$f_y = (x)(x+2y)^{-1} + (xy)(-1)(x+2y)^{-2}(2)$$

$$f_y(1,2) = (1)(5)^{-1} + (-4)(5)^{-2} = \frac{1}{5} - \frac{4}{25} = \frac{1}{25}$$

Tangent plane:

$$L(x,y) = \boxed{\frac{2}{5} + \frac{8}{25}(x-1) + \frac{1}{25}(y-2)}$$

$$= \frac{2}{5} + \frac{8}{25}x - \frac{8}{25} + \frac{1}{25}y - \frac{2}{25} = \frac{8}{25}x + \frac{1}{25}y$$

Normal vector to plane: $\left(\frac{-8}{25}, \frac{-1}{25}, 1\right)$

2. (15 pts.) Find the directional derivative of the function $f(x,y) = xy^2 + x^2y$ in the direction of the velocity vector of the parametrized curve $\gamma(t) = (t\sin(t), 2-t)$, at time $t = \pi/2$.

$$\nabla f = (f_x, f_y) = (y^2 + 2xy,\ 2xy + x^2)$$

$$\gamma'(t) = (\sin t + t\cos t,\ -1)$$

$$\gamma\left(\tfrac{\pi}{2}\right) = \left(\tfrac{\pi}{2}\sin\tfrac{\pi}{2},\ 2-\tfrac{\pi}{2}\right) = \left(\tfrac{\pi}{2},\ 2-\tfrac{\pi}{2}\right)$$

$$\gamma'\left(\tfrac{\pi}{2}\right) = \left(\sin\tfrac{\pi}{2} + \tfrac{\pi}{2}\cos\tfrac{\pi}{2},\ -1\right) = (1,-1) = \vec{v}$$

$$\nabla f\left(\gamma\left(\tfrac{\pi}{2}\right)\right) = \nabla f\left(\tfrac{\pi}{2},\ 2-\tfrac{\pi}{2}\right)$$

$$= \left(\left(2-\tfrac{\pi}{2}\right)^2 + 2\tfrac{\pi}{2}\left(2-\tfrac{\pi}{2}\right),\ 2\tfrac{\pi}{2}\left(2-\tfrac{\pi}{2}\right) + \left(\tfrac{\pi}{2}\right)^2\right)$$

Directional derivative $= D_v f\left(\gamma\left(\tfrac{\pi}{2}\right)\right)$

$$= \nabla f\left(\gamma\left(\tfrac{\pi}{2}\right)\right) \cdot \gamma'\left(\tfrac{\pi}{2}\right)$$

$$= \left[\left(2-\tfrac{\pi}{2}\right)^2 + \cancel{2\tfrac{\pi}{2}\left(2-\tfrac{\pi}{2}\right)}\right](1) + \left[\cancel{2\tfrac{\pi}{2}\left(2-\tfrac{\pi}{2}\right)} + \left(\tfrac{\pi}{2}\right)^2\right](-1)$$

$$= \left(2-\tfrac{\pi}{2}\right)^2 - \left(\tfrac{\pi}{2}\right)^2$$

$$= 4 - 2\cdot 2\cdot\tfrac{\pi}{2} + \cancel{\left(\tfrac{\pi}{2}\right)^2} - \cancel{\left(\tfrac{\pi}{2}\right)^2} = 4 - 2\pi$$

3. (20 pts.) Recall that the line $y = L(x) = ax + b$ that 'best fits' a collection (x_i, y_i) of points is the one which minimizes the quantity $\sum_{i=1}^{n}(L(x_i) - y_i)^2$. Find the best fitting line for the points

$$(0,0)\ ,\ (1,2)\ ,\ \text{and}\ (3,2)\ .$$

$L(x) = ax+b$

minimize

$$((a\cdot 0+b)-0)^2 + ((a\cdot 1+b)-2)^2 + ((3a+b)-2)^2 = f(a,b)$$

$$= b^2 + (a+b-2)^2 + (3a+b-2)^2$$

$$= b^2 + (a+b)^2 - 2(a+b)(2) + 2^2 + (3a+b)^2 - 2(3a+b)(2) + 2^2$$

$$= b^2 + a^2 + 2ab + b^2 - 4a - 4b + 4 + 9a^2 + 6ab + b^2 - \cancel{6a}\,12a - 4b + 4$$

$$= 10a^2 + 8ab + 3b^2 - 16a - 8b + 8$$

$$f_a = 20a + 8b + 0 - 16 + 0 + 0 = 20a + 8b - 16 = 0$$

$$f_b = 0 + 8a + 6b - 0 - 8 + 0 = 8a + 6b - 8 = 0$$

So need $20a + 8b = 16 \longrightarrow 5a + 2b = 4$

$8a + 6b = 8 \qquad 4a + 3b = 4$

$a - b = 0 \longrightarrow \boxed{a = b}$

$5a + 2b = 5a + 2a = 7a = 4$

$a = 4/7 \longrightarrow b = 4/7$

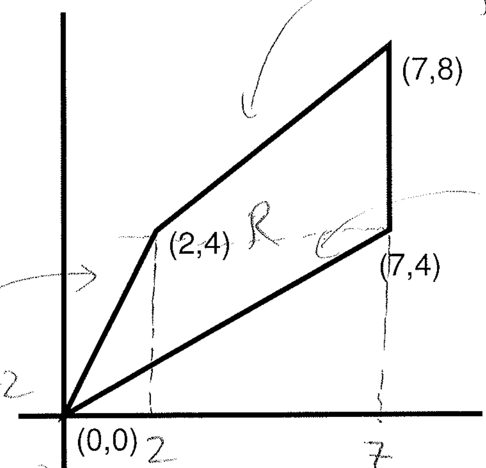
$\boxed{L(x) = \frac{4}{7}x + \frac{4}{7}}$ gives best fit.

4. (15 pts.) Show how to express a double integral of some function $z = f(x, y)$ over the region R lying inside of the polygon shown below, as a sum of one or more iterated integrals.

slope $= \frac{8-4}{7-2} = \frac{4}{5}$ $\quad y = 4 + \frac{4}{5}(x-2)$

$= 4 + \frac{4}{5}x - \frac{8}{5} = \frac{4}{5}x + \frac{12}{5}$

slope $= \frac{4-0}{7-0} = \frac{4}{7}$ $\quad y = 0 + \frac{4}{7}(x-0) = \frac{4}{7}x$



slope $= \frac{4-0}{2-0} = 2$

$y = 0 + 2(x-0)$
$= 2x$

$\frac{4}{7}x \le y \le 2x$ for $0 \le x \le 2$ and $\frac{4}{7}x \le y \le \frac{4}{5}x + \frac{12}{5}$ for $2 \le x \le 7$

$$\int_R f(x,y)\,dA = \int_0^2 \int_{\frac{4}{7}x}^{2x} f(x,y)\,dy\,dx + \int_2^7 \int_{\frac{4}{7}x}^{\frac{4}{5}x+\frac{12}{5}} f(x,y)\,dy\,dx$$

or

$\frac{y}{2} \le x \le \frac{7}{4}y$ for $0 \le y \le 4$ and $\frac{5}{4}\left(y - \frac{12}{5}\right) \le x \le 7$ for $4 \le y \le 8$

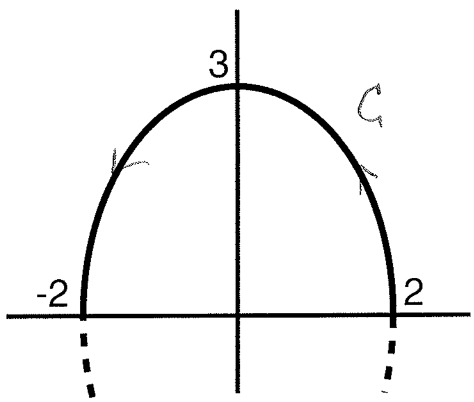
$$\int_R f(x,y)\,dA = \int_0^4 \int_{\frac{1}{2}y}^{\frac{7}{4}y} f(x,y)\,dx\,dy + \int_4^8 \int_{\frac{5}{4}y-3}^{7} f(x,y)\,dx\,dy$$

5. (20 pts.) Find the work done by the vector field

$$\vec{F}(x, y) = (1, x^2)$$

along the top half of the ellipse given by $\left(\frac{x}{2}\right)^2 + \left(\frac{y}{3}\right)^2 = 1$, from $(2, 0)$ to $(-2, 0)$ (see figure).

$\cancel{\sin}^2 t + \cancel{\cos}^2 t = 1$
$\cos \qquad \sin$



$\frac{x}{2} = \cancel{\sin}\cos t \qquad x = 2\cos t$

$\frac{y}{3} = \sin t \qquad y = 3\sin t$

for $0 \le t \le \pi$

$\gamma(t) = (2\cos t, 3\sin t)$

$\gamma'(t) = (-2\sin t, 3\cos t)$

$F(\gamma(t)) = (1, (2\cos t)^2) = (1, 4\cos^2 t)$

$\int_C \vec{F} \cdot d\vec{r} = \int_0^\pi (1, 4\cos^2 t) \cdot (-2\sin t, 3\cos t)\, dt$

$= \int_0^\pi -2\sin t + 12\cos^3 t\, dt = \int_0^\pi -2\sin t + 12(1 - \sin^2 t)\cos t\, dt$

$= 2\cos t + 12\left(\sin t - \frac{\sin^3 t}{3}\right)\Big|_0^\pi$

$= 2(\cos\pi - \cos 0) + 12\left((\sin\pi - \sin 0) - \frac{1}{3}(\sin^3\pi - \sin^3 0)\right)$

$= 2(-1 - (1)) + 12\left((0 - 0) - \frac{1}{3}(0 - 0)\right) = 2(-2) = \boxed{-4}$

6. (15 pts.) Show that the vector field $\vec{F}(x,y) = (y + \frac{1}{x}, x + \frac{1}{y})$ is a conservative vector field, and find a potential function for $\vec{F}$.

$F_1 = y + \frac{1}{x} \qquad F_2 = x + \frac{1}{y}$

$(F_2)_x = 1 = (F_1)_y \quad$ so $\quad \text{curl}(\vec{F}) = 0$

$f(x,y) = \int F_1(x,y)\,dx = \int y + \frac{1}{x}\,dx = xy + \ln(x) + C(y)$

$x + \frac{1}{y} = f_y(x,y) = x + 0 + C'(y)$

$\rightarrow C'(y) = \frac{1}{y}, \quad C(y) = \int \frac{1}{y}\,dy = \ln(y) + C$

so $f(x,y) = xy + \ln(x) + \ln(y) + C$

is a potential function for $\vec{F}$

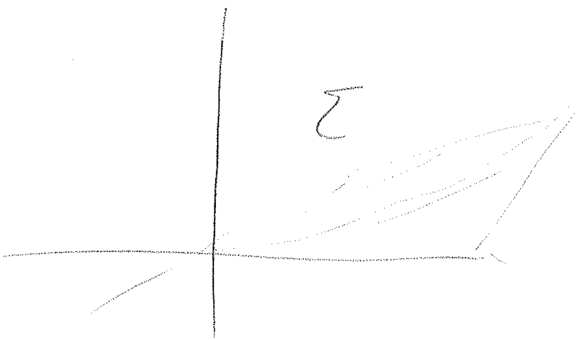
<u>Check!</u> $\quad f_x = y + \frac{1}{x} + 0 = y + \frac{1}{x} = F_1$ ✓

$f_y = x + 0 + \frac{1}{y} = x + \frac{1}{y} = F_2$ ✓

7. (20 pts.) Find the flux of the vector field

$$\vec{F}(x,y,z) = (y, y, yz)$$

through the graph of the function $z = f(x,y) = xy$ which lies above the rectangular region R in the plane lying between the x- and y-axes and the lines $x = 3$, $y = 2$.



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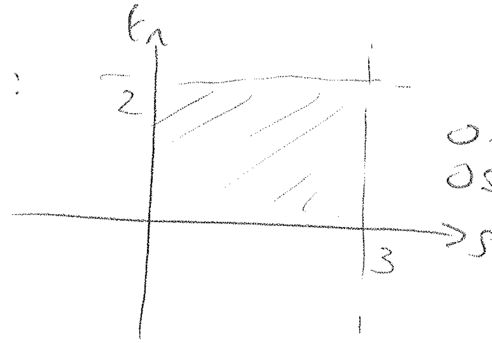


$0 \le s \le 3$
$0 \le t \le 2$

$$S(s,t) = (s, t, st)$$

$$\frac{\partial S}{\partial s} = (1, 0, t), \quad \frac{\partial S}{\partial t} = (0, 1, s)$$

$$\frac{\partial S}{\partial s} \times \frac{\partial S}{\partial t} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 0 & t \\ 0 & 1 & s \end{vmatrix} = \vec{i}(-t) - \vec{j}(s) + \vec{k}(1)$$
$$= (-t, -s, 1)$$

$$\vec{F}(S(s,t)) = (t, t, st^2)$$

$$\int_\Sigma \vec{F} \cdot d\vec{A} = \int_0^2 \int_0^3 (t, t, st^2) \cdot (-t, -s, 1)\, ds\, dt$$

$$= \int_0^2 \int_0^3 -t^2 - st + st^2 \, ds\, dt = \int_0^2 -st^2 - \frac{s^2}{2} t + \frac{s^2}{2} t^2 \Big|_0^3 dt$$

$$= \int_0^2 \left(-3t^2 - \frac{9}{2} t + \frac{9}{2} t^2\right) - 0 \, dt = \int_0^2 \frac{3}{2} t^2 - \frac{9}{2} t \, dt$$

$$= \frac{1}{2} t^3 - \frac{9}{4} t^2 \Big|_0^2 = \frac{1}{2}(8) - \frac{9}{4}(4) - 0 = 4 - 9 = \boxed{-5}$$

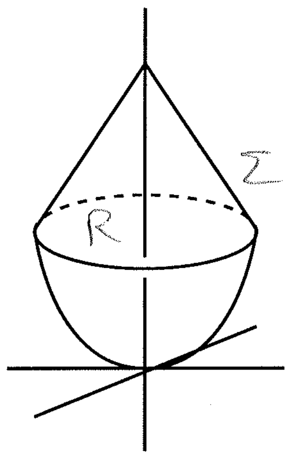
8. (15 pts.) Use the Divergence Theorem to **set up but not evaluate** the integral required to find the flux of the vector field

$$\vec{F}(x,y,z) = (x, 2, xz)$$

through the boundary of the region lying between the graphs of the functions

$$f(x,y) = x^2 + y^2 \qquad \text{and} \qquad g(x,y) = 6 - \sqrt{x^2+y^2} \qquad \text{(see figure!).}$$

[Hint: to find out where the graphs meet, set $r = \sqrt{x^2+y^2}$ and solve for r...]



$$\int_\Sigma \vec{F} \cdot d\vec{A} = \int_R \operatorname{div}(\vec{F})\, dV$$

$$\operatorname{div}(\vec{F}) = (x)_x + (2)_y + (xz)_z = 1 + x$$

Cylindrical coords!

$$x^2+y^2 \le z \le 6 - \sqrt{x^2+y^2}$$

$$\to \boxed{r^2 \le z \le 6 - r}$$

Shadow? $r^2 = 6 - r$ $\qquad r^2 + r - 6 = 0 = (r+3)(r-2)$

$\to \cancel{r=-3}$ or $\boxed{r=2}$ circle of radius 2

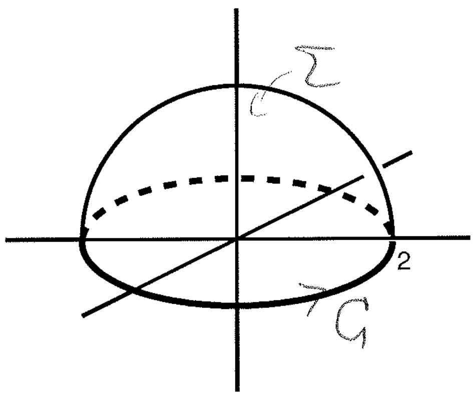
R: $\boxed{\begin{array}{l} r^2 \le z \le 6-r \\ 0 \le r \le 2 \\ 0 \le \theta \le 2\pi \end{array}}$ $\qquad 1 + x = 1 + r\cos\theta$

$$\int_\Sigma \vec{F} \cdot d\vec{A} = \boxed{\int_0^{2\pi} \int_0^2 \int_{r^2}^{6-r} (1 + r\cos\theta)\, r\, dz\, dr\, d\theta}$$

or

$$\int_{-2}^{2} \int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} \int_{x^2+y^2}^{6-\sqrt{x^2+y^2}} 1 + x \; dz\, dy\, dx$$

9. (20 pts.) Use the fact that $\vec{F}(x,y,z) = (1, xy, 1-xz) = \text{curl}(xyz, x, y)$ to use Stokes' Theorem to compute the flux integral of $\vec{F}$ over the top half of the sphere of radius 2 centered at the origin, $\{(x,y,z)\ :\ x^2+y^2+z^2=4, z\geq 0\}$ (see figure).



$$\vec{F} = \text{curl}(xyz, x, y) = \text{curl}(\vec{G})$$

$$\int_\Sigma \vec{F}\cdot d\vec{A} = \int_\Sigma \text{curl}(\vec{G})\cdot d\vec{A}$$

$$= \int_G \vec{G}\cdot d\vec{r} \quad \text{(by Stokes)}$$

G = circle of radius 2

$$\gamma(t) = (2\cos t, 2\sin t, 0) \qquad 0 \leq t \leq 2\pi$$

$$\vec{G}(\gamma(t)) = \vec{G}(2\cos t, 2\sin t, 0)$$

$$= ((2\cos t)(2\sin t)(0), 2\cos t, 2\sin t) = (0, 2\cos t, 2\sin t)$$

$$\gamma'(t) = (-2\sin t, 2\cos t, 0)$$

$$\int_G \vec{G}\cdot d\vec{r} = \int_0^{2\pi} (0, 2\cos t, 2\sin t)\cdot(-2\sin t, 2\cos t, 0)\, dt$$

$$= \int_0^{2\pi} 4\cos^2 t\, dt = 4\left(\frac{1}{2}(t + \sin t\cos t)\right)\Big|_0^{2\pi}$$

$$= 2\left[(2\pi + \sin 2\pi \cos 2\pi) - (0 + \sin 0 \cos 0)\right]$$

$$= 2[2\pi + 0 - 0] = \boxed{4\pi}$$

Some potentially useful formulas:

Spherical coordinates:

$$x = \rho \cos\theta \sin\phi$$
$$y = \rho \sin\theta \sin\phi$$
$$z = \rho \cos\phi$$

Cylindrical coordinates:

$$x = r\cos\theta$$
$$y = r\sin\theta$$
$$z = z$$

$$\cos(A)\cos(B) = \frac{1}{2}[\cos(A+B) + \cos(A-B)]$$
$$\sin(A)\sin(B) = \frac{1}{2}[\cos(A-B) - \cos(A+B)]$$
$$\sin(A)\cos(B) = \frac{1}{2}[\sin(A+B) + \sin(A-B)]$$
$$\cos(2x) = \cos^2 x - \sin^2 x$$
$$\sin(2x) = 2\sin x \cos x$$
$$\int \cos^2 x \, dx = \frac{1}{2}(x + \sin x \cos x) + C$$
$$\int \sin^2 x \, dx = \frac{1}{2}(x - \sin x \cos x) + C$$