Math 221, Section 5

Quiz number 4 Solutions

Show all work. How you get your answer is just as important, if not more important, than the answer itself. If you think it, write it!

1. Find the equilibrium solutions to the autonomous differential equation

$$y' = \frac{y(3-y)^2}{y^2+1}$$

and determine whether each is stable, semistable, or unstable. If we were to find the solution to this equation having initial value $y(0) = 4$, what will be the limit of $y(x)$ as x approaches ∞ ?

To find the equilibrium (= constant) solutions, we find the roots

$$f(y) = \frac{y(3-y)^2}{y^2+1} = 0$$

This occurs when $y(3-y)^2 = 0$, i.e., $y = 0$ or $3 - y = 0$, so $y = 3$. So the equilibrium solutions are $y = 0$ and $y = 3$.

To determine the stability of each equilibrium, since $f(y)$ is continuous (it is a rational function whose denomenator is never 0), it suffices to determine the sign of f in each of the intervals between the equilibria. Since the denomenator of f, $y^2 + 1$, is always positive, it will play no role in these considerations. Plugging in,

$$f(-1) = (-1)(4)^2/2 = -2 < 0,$$
$$f(1) = (1)(2)^2/2 = 2 > 0, \text{ and}$$
$$f(4) = (4)(-1)^2/17 = 4/17 > 0,$$

so solutions with initial values in $(-\infty, 0)$ are decreasing while solutions starting in $(0, 3)$ and $(3, \infty)$ are increasing.

So solutions on either side of $y = 0$ move away from 0 as x increases, so $y = 0$ is an unstable equilibrium. Solutions just below $y = 3$ move toward 3 as x increases, but solutions above $y = 3$ move away from 3, so $y = 3$ is a semistable equilibrium.

Since the solution to our differential equation with $y(0) = 4$ will be increasing, and there is no equilibrium solution lying above $y = 4$, the solution will continue to increase without bound, and $\lim_{x\to\infty} y(x) = \infty$.