Math 107H

Topics for the first exam

Integration

Basic list:

$\int x^n \, \mathrm{d}x = \dfrac{x^{n+1}}{n+1} + C$ (provided $n \neq -1$) $\quad \int 1/x \, \mathrm{d}x = \ln|x| + C$

$\int \sin(kx) \, \mathrm{d}x = \frac{-\cos(kx)}{k} + C \quad \int \cos(kx) \, \mathrm{d}x = \frac{\sin(kx)}{k} + C$

$\int \sec^2 x \, \mathrm{d}x = \tan x + C \quad \int \csc^2 x \, \mathrm{d}x = -\cot x + C$

$\int \sec x \tan x \, \mathrm{d}x = \sec x + C \quad \int \csc x \cot x \, \mathrm{d}x = -\csc x + C$

$\int \tan x \, \mathrm{d}x = \ln|\sec x| + C \quad \int \sec x \, \mathrm{d}x = \ln|\sec x + \tan x| + C$

$\int \cot x \, \mathrm{d}x = \ln|\sin x| + C \quad \int \csc x \, \mathrm{d}x = -\ln|\csc x + \cot x| + C$

$\int e^x \, \mathrm{d}x = e^x + C \quad \int \frac{dx}{\sqrt{a^2-x^2}} = \mathrm{Arcsin}(\frac{x}{a}) + c$

$\int \frac{dx}{x^2+a^2} = \frac{1}{a} \mathrm{Arctan}(\frac{x}{a}) + c \quad \int \frac{dx}{|x|\sqrt{x^2-a^2}} = \frac{1}{a}\mathrm{Arcsec}(\frac{x}{a}) + c$

Basic integration rules: for k=constant,

$\int k \cdot f(x) \, \mathrm{d}x = k \int f(x) \, \mathrm{d}x \quad \int (f(x) \pm g(x) \, \mathrm{d}x = \int f(x) \, \mathrm{d}x \pm \int g(x) \, \mathrm{d}x$

The Fundamental Theorem of Calculus

$\int_a^x f(t) \, dt = F(x)$ is a function of x. $\quad F(x) =$ the area under graph of f, from a to x.

FTC 2: If f is cts, then $F'(x) = f(x)$ $\quad$ (F is an antideriv of f !)

Since any two antiderivatives differ by a constant, and $F(b) = \int_a^b f(t) \, dt$, we get

FTC 1: If f is cts, and F is an antideriv of f, then $\int_a^b f(x) \, dx = F(b) - F(a) = F(x) \mid_a^b$

Integration by substitution. The idea: reverse the chain rule!

$g(x) = u$, then $\dfrac{d}{dx}f(g(x))=\dfrac{d}{dx}f(u) = f'(u) \, \dfrac{du}{dx}$, so $\int f'(u) \, \dfrac{du}{dx} \, dx = \int f'(u) \, du = f(u) + c$

$\int f(g(x))g'(x) \, dx$; set $u = g(x)$, then $du = g'(x) \, dx$,

so $\int f(g(x))g'(x) \, dx = \int f(u) \, du$, <u>where</u> $u = g(x)$

Example: $\int x(x^2-3)^4 \, dx$; set $u = x^2 - 3$, so du=$2x \, dx$. Then

$\int x(x^2-3)^4 \, dx = \frac{1}{2}\int (x^2-3)^4 2x \, dx = \frac{1}{2}\int u^4 \, du \mid_{u=x^2-3} = \frac{1}{2}\frac{u^5}{5} + c \mid_{u=x^2-3} = \frac{(x^2-3)^5}{10} + c$

The three most important points:

1. Make sure that you calculate (and then set aside) your du before doing step 2!
2. Make sure everything gets changed from x's to u's
3. **<u>Don't</u>** push x's through the integral sign! They're <u>not</u> constants!

We can use u-substitution directly with a definite integral, provided we remember that

$\displaystyle\int_a^b f(x) \, dx$ <u>really</u> means $\displaystyle\int_{x=a}^{x=b} f(x) \, dx$, and we remember to change <u>all</u> x's to u's!

Ex: $\displaystyle\int_1^2 x(1+x^2)^6 \, dx$; set $u = 1 + x^2$, $du = 2x \, dx$. when $x = 1$, $u = 2$; when $x = 2$, $u = 5$;

so $\displaystyle\int_1^2 x(1+x^2)^6 \, dx = \frac{1}{2}\int_2^5 u^6 \, du = ...$

Basic integration formulas (AKA dirty tricks)

complete the square

$ax^2 + bx + c = a(x^2 + rx) + c = a(x + r/2)^2 + (c - (r/2)^2)$

Ex: $\displaystyle\int \frac{1}{x^2 + 2x + 2}\ \mathrm{d}x = \int \frac{1}{(x+1)^2 + 1}\ \mathrm{d}x$

use trig identities

$\sin^2 x + \cos^2 x = 1$, $\tan^2 x + 1 = \sec^2 x$, $\sin(2x) = 2\sin x \cos x$, $\dfrac{\tan x}{\sec x}$=$\sin x$, etc.

Ex: $\displaystyle\int \frac{\sin^2 x}{\cos x}\ \mathrm{d}x = \int \frac{1 - \cos^2 x}{\cos x}\ \mathrm{d}x = \ldots$

pull fractions apart; put fractions together!

Ex: $\displaystyle\int \frac{x+1}{x^3}\ \mathrm{d}x = \int x^{-2} + x^{-3}\ \mathrm{d}x = \ldots$

do polynomial long division

Ex: $\displaystyle\int \frac{x^3}{x^2 - 1}\ \mathrm{d}x = \int x + \frac{x}{x^2 - 1}\ \mathrm{d}x = \ldots$

add zero, multiply by one

Ex: $\displaystyle\int \sec x\ \mathrm{d}x = \int \frac{\sec x(\tan x + \sec x)}{\sec x + \tan x}\ \mathrm{d}x = \ldots$

Integration by parts

Product rule: $\mathrm{d}(uv) = (\mathrm{d}u)v + u(\mathrm{d}v)$

reverse: $\int u\ \mathrm{d}v = uv - \int v\ \mathrm{d}u$

Ex: $\int x \cos x\ \mathrm{d}x$: set u=x, $\mathrm{d}v$=$\cos x\ \mathrm{d}x$ $\quad$ $\mathrm{d}u$=$\mathrm{d}x$, $v = \sin x$ (or any <u>other</u> antiderivative)

So: $\int x \cos x = x \sin x - \int \sin x\ \mathrm{d}x = \ldots$

special case: $\displaystyle\int f(x)\ \mathrm{d}x$; $u = f(x)$, $\mathrm{d}v$=$\mathrm{d}x$ $\quad \displaystyle\int f(x)\ \mathrm{d}x = xf(x) - \int xf'(x)\ \mathrm{d}x$

Ex: $\displaystyle\int \text{Arcsin } x\ \mathrm{d}x = x \text{ Arcsin } x - \int \frac{x}{\sqrt{1 - x^2}} = \ldots$

The basic idea: integrate part of the function (a part that you <u>can</u>), differentiate the rest. Goal: reach an integral that is "nicer".

Ex: $\int x^3 \ln x\ dx = (x^4/4) \ln x - \int (x^4/4)(1/x)\ dx = \ldots$

Trig substitution

Idea: get rid of square roots, by turning the stuff inside into a perfect square!

$\sqrt{a^2 - x^2}$: set $x = a \sin u$. $\mathrm{d}x = a \cos u\ \mathrm{d}u$, $\sqrt{a^2 - x^2} = a \cos u$

Ex: $\displaystyle\int \frac{1}{x^2\sqrt{1 - x^2}}\ \mathrm{d}x = \int \frac{\cos u}{\sin^2 u \cos u}\ \mathrm{d}u\Big|_{x=\sin u} = \ldots$

$\sqrt{a^2 + x^2}$: set $x = a \tan u$. $\mathrm{d}x = a \sec^2 u\ \mathrm{d}u$, $\sqrt{a^2 + x^2} = a \sec u$

Ex: $\displaystyle\int \frac{1}{(x^2 + 4)^{3/2}}\ \mathrm{d}x = \int \frac{2 \sec^2 u}{(2 \sec u)^3}\ \mathrm{d}u\Big|_{x=2\tan u} = \ldots$

$\sqrt{x^2 - a^2}$: set $x = a \sec u$. $\mathrm{d}x = a \sec u \tan u\ \mathrm{d}u$, $\sqrt{x^2 - a^2} = a \tan u$

Ex: $\displaystyle\int \frac{1}{x^2\sqrt{x^2-1}}\,\mathrm{d}x = \int \frac{\sec u\tan u}{\sec^2 u\tan u}\,\mathrm{d}u\Big|_{x=\sec u} = \dots$

Undoing the "u-substitution": use right triangles! (Draw a right triangle!)
Ex: $x = a\sin u$, then angle u has opposite $= x$, hypotenuse $= a$, so adjacent $= \sqrt{a^2-x^2}$. So $\cos u = (\sqrt{a^2-x^2})/a$, $\tan u = x/\sqrt{a^2-x^2}$, etc.

Trig integrals: What trig substitution usually leads to!

$\displaystyle\int \sin^n x\ \cos^m x\ dx$

If n is odd, keep one $\sin x$ and turn the others, in pairs, into $\cos x$
(using $\sin^2 x = 1-\cos^2 x$), then do a u-substitution $u = \cos x$.
If m is odd, reverse the roles of $\sin x$ and $\cos x$.
If both are even, turn the $\sin x$ into $\cos x$ (in pairs) and use the double angle formula

$$\cos^2 x = \frac{1}{2}(1+\cos(2x))$$

This will convert $\cos^m x$ into a bunch of *lower powers* of $\cos(2x)$;
odd powers can be dealt with by substitution, even powers by another application of the angle doubling formula!

$\displaystyle\int \sec^n x\ \tan^m x\ dx = \int \frac{\sin^m x}{\cos^{n+m} x}\ dx$

If n is *even*, set two of them aside and convert the rest to $\tan x$
using $\sec^2 x = \tan^2 x + 1$, and use $u = \tan x$.
If m is *odd*, set one each of $\sec x$, $\tan x$ aside, convert the rest of the $\tan x$ to $\sec x$
using $\tan^2 x = \sec^2 x - 1$, and use $u = \sec x$.

If n is odd and m is even, convert all of the $\tan x$ to $\sec x$ (in pairs),
leaving a bunch of powers of $\sec x$. Then use the *reduction formula*:

$$\int \sec^n x\ dx = \frac{1}{n-1}\sec^{n-2} x\tan x + \frac{n-2}{n-1}\int \sec^{n-2} x\ dx$$

At the end, reach $\int sec^2 x\ dx = \tan x + C$ or $\int \sec x\ dx = \ln|\sec x+\tan x| + C$

A little "trick" worth knowing:
the substitution $u = \frac{\pi}{2} - x$, since $\sin(\frac{\pi}{2}-x) = \cos x$ and $\cos(\frac{\pi}{2}-x) = \sin x$,
will *reverse* the roles of $\sin x$ and $\cos x$,
so will turn $\cot x$ into $\tan u$ and $\csc x$ into $\sec u$. So, for example, the integral
$\displaystyle\int \frac{\cos^4 x}{\sin^7 x}\ dx = \int \csc^3 x\cot^4 x\ dx$, which our techniques don't cover,

becomes $\displaystyle\int \sec^3 u\tan^4 u\ du$, which our techniques do cover.

Partial fractions

rational function = quotient of polynomials

Idea: integrate by writing function as sum of simpler functions

Procedure: $f(x) = p(x)/q(x)$

(0): arrange for degree(p)<degree(q); do long division if it isn't

(1): factor $q(x)$ into linear and irreducible quadratic factors

(2): group common factors together as powers

(3a): for each group $(x-a)^n$ add together: $\frac{a_1}{x-a} + \cdots + \frac{a_n}{(x-a)^n}$

(3b): for each group $(ax^2+bx+c)^n$ add together:

$$\frac{a_1x+b_1}{ax^2+bx+c} + \cdots + \frac{a_nx+b_n}{(ax^2+bx+c)^n}$$

(4) set $f(x)$ = sum of all sums; solve for the 'undetermined' coefficients

put sum over a common denomenator (=$q(x)$); set numerators equal.

always works: multiply out, group common powers, set coeffs of the two polys equal

Ex: $x+3 = a(x-1)+b(x-2) = (a+b)x+(-a-2b)$; $1 = a+b$, $3 = -a-2b$

linear term $(x-a)^n$: set $x = a$, will allow you to solve for a coefficient

if $n \geq 2$, take derivatives of both sides! set x=a, gives another coeff.

Ex: $$\frac{x^2}{(x-1)^2(x^2+1)} = \frac{A}{x-1} + \frac{B}{(x-1)^2} + \frac{Cx+D}{x^2+1}$$
$$= \frac{A(x-1)(x^2+1)+B(x^2+1)+(Cx+D)(x-1)^2}{(x-1)^2(x^2+1)} = \ldots$$

Improper integrals

Fund Thm of Calc: $\int_a^b f(x)\ \mathrm{d}x = F(b)-F(a)$, where $F'(x) = f(x)$

Problems: $a = -\infty$, $b = \infty$; f blows up at a or b or somewhere in between

integral is"improper"; usual technique doesn't work. Solution to this:

$$\int_a^\infty f(x)\ \mathrm{d}x = \lim_{b\to\infty}\int_a^b f(x)\ \mathrm{d}x \qquad \int_{-\infty}^b f(x)\ \mathrm{d}x = \lim_{a\to-\infty}\int_a^b f(x)\ \mathrm{d}x$$

(blow up at a) $\int_a^b f(x)\ \mathrm{d}x = \lim_{r\to a^+}\int_r^b f(x)\ \mathrm{d}x = \lim_{\epsilon\to 0^+}\int_{a+\epsilon}^b f(x)\ \mathrm{d}x$

(similarly for blowup at b (or both!))

$$\int_a^b f(x)\ \mathrm{d}x = \lim_{s\to b^-}\int_a^s f(x)\ \mathrm{d}x = \lim_{\epsilon\to 0^+}\int_a^{b-\epsilon} f(x)\ \mathrm{d}x$$

(blows up at c (b/w a and b)) $\int_a^b f(x)\ \mathrm{d}x = \lim_{r\to c^-}\int_a^r f(x)\ \mathrm{d}x + \lim_{s\to c^+}\int_s^b f(x)\ \mathrm{d}x$

The integral <u>converges</u> if (all of the) limit(s) are finite

Comparison: $0 \leq f(x) \leq g(x)$ for all x;

if $\int_a^\infty g(x)\ \mathrm{d}x$ *converges*, so does $\int_a^\infty f(x)\ \mathrm{d}x$

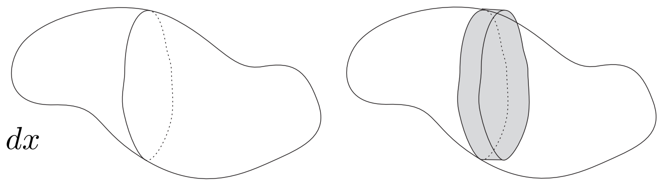
if $\int_a^\infty f(x)\ \mathrm{d}x$ *diverges*, so does $\int_a^\infty g(x)\ \mathrm{d}x$

Applications of integration

Volume by slicing. To calculate volume, aprroximate region by objects whose volume we can calculate.

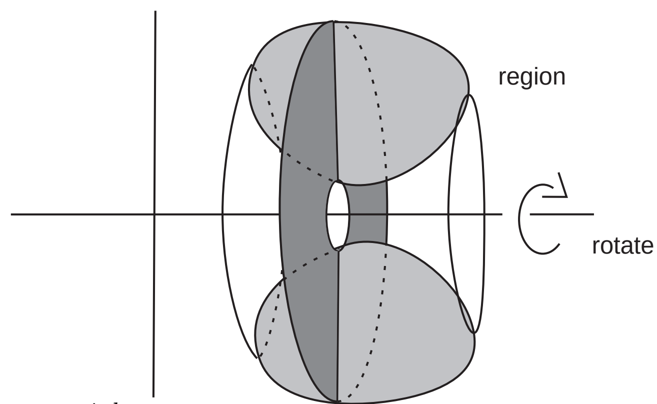
Volume $\approx \sum$(volumes of 'cylinders')
$= \sum$(area of base)(height)
$= \sum$(area of cross-section)Δx_i .

So volume $= \displaystyle\int_{left}^{right}$ (area of cross section) dx



Solids of revolution: disks and washers. Solid of revolution: take a region in the plane and revolve it around an axis in the plane.

take cross-sections perpendicular to axis of revolution;
cross-section = disk (area=πr^2)
or washer (area=$\pi R^2 - \pi r^2$)
rotate around x-axis: write r (and R) as functions of x, integrate dx
rotate around y-axis: write r (and R) as functions of y, integrate dy



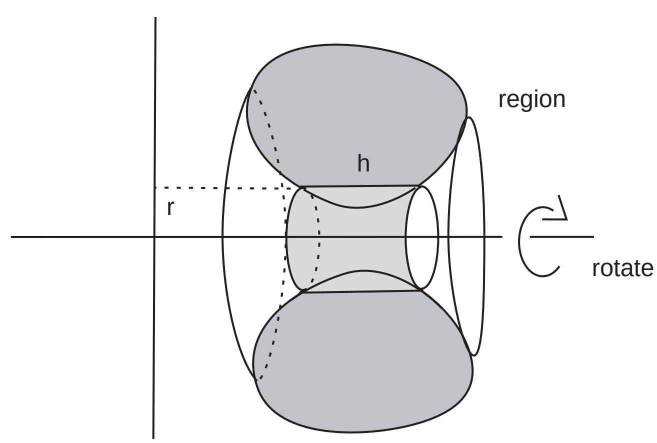
Otherwise, everything is as before: volume $= \displaystyle\int_{left}^{right} A(x)\,dx$ or volume $= \displaystyle\int_{bottom}^{top} A(y)\,dy$

The same is true if axis is parallel to $x-$ or $y-$axis; r and R just change (we add a constant).

Cylindrical shells. Different picture, same volume! Solid of revolution; use cylinders centered on the axis of revolution. The intersection is a cylinder, with area = (circumference)(height) = $2\pi r h$

volume $= \displaystyle\int_{left}^{right}$ (area of cylinder) dx or $\displaystyle\int_{bottom}^{top}$ (area of cylinder) dy!)

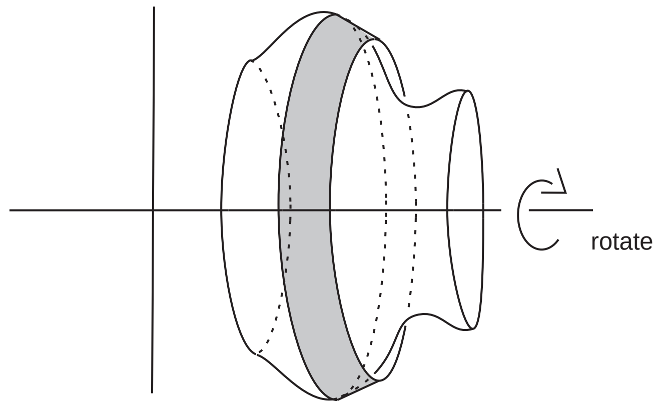
revolve around vertical line: integrate dx
revolve around horizontal line: integrate dy



Ex: region in plane between $y = 4x$, $y = x^2$, revolved around y-axis

left=0, right=4, $r = x$, $h = (4x - x^2)$ volume $= \int_0^4 2\pi x(4x - x^2)\ dx$

Arclength and surface area



Arclength. Idea: approximate a curve by lots of short line segments; length of curve ≈ sum of lengths of line segments.

Line segment between $(c_i, f(c_i))$ and $(c_{i+1}, f(c_{i+1}))$:

$$\sqrt{1 + (\frac{f(c_{i+1}) - f(c_i)}{c_{i+1} - c_i})^2} \cdot (c_{i+1} - c_i) \approx \sqrt{1 + (f'(c_i))^2} \cdot \Delta x_i$$

So length of curve $= \int_{left}^{right} \sqrt{1 + (f'(x))^2}\ dx$

The problem: integrating $\sqrt{1 + (f'(x))^2}$! Sometimes, $1 + (f'(x))^2$ turns out to be a perfect square.....

Surface area. Idea: find the area of a surface (of revolution) by approximating the surface by things whose area we can figure out. Frustrum of a cone!

area of frustrum $= \pi \cdot \big(f(c_{i+1}) + f(c_i)\big) \cdot \sqrt{1 + \Big(\frac{f(c_{i+1}) - f(c_i)}{c_{i+1} - c_i}\Big)^2} \cdot (c_{i+1} - c_i)$

$\approx 2\pi f(c_i \cdot \sqrt{1 + (f'(c_i))^2} \cdot \Delta x_i$. So area of surface $= \int_{left}^{right} 2\pi f(x)\sqrt{1 + (f'(x))^2}\ dx$

The problem: same problem! But sometimes it's possible to do.... Ex: for $f(x) = \sqrt{r^2 - x^2}$, the thing to integrate simplifies to: $2\pi r$!