Math 107H
Topics for the second exam

(Technically, everything covered on the first exam plus...)

Infinite sequences and series

Limits of sequences of numbers

A sequence is: a string of numbers; a function $f:\mathbf{N}\to\mathbf{R}$; write $f(n) = a_n$
$a_n = n$-th term of the sequence

Basic question: convergence/divergence
$\lim_{n\to\infty} a_n = L$ (or $a_n \to L$) if
eventually all of the a_n are always as close to L as we like, i.e.
for any $\epsilon > 0$, there is an N so that if $n \geq N$ then $|a_n - L| < \epsilon$
Ex.: $a_n = 1/n$ converges to 0 ; can always choose N=$1/\epsilon$
$a_n = (-1)^n$ diverges; terms of the sequence never settle down to a single number

If a_n is increasing ($a_{n+1} \geq a_n$ for every n) and bounded from above
($a_n \leq M$ for every n, for some M) , then a_n converges (but not necessarily to M !)
limit is smallest number bigger than all of the terms of the sequence

Limit theorems for sequences

Idea: limits of sequences are a lot like limits of functions

If $a_n \to L$ and $b_n \to M$, then
$(a_n + b_n \to L + M$ $\quad (a_n - b_n) \to L - M \quad (a_n b_n) \to LM$, and
$(a_n/b_n) \to L/M$ (provided M, all b_n are $\neq 0$)

Sqeeze play theorem: if $a_n \leq b_n \leq c_n$ (for all n large enough) and
$a_n \to L$ and $c_n \to L$, then $b_n \to L$

If $a_n \to L$ and $f:\mathbf{R}\to\mathbf{R}$ is continuous at L, then $f(a_n) \to f(L)$

if $a_n = f(n)$ for some function $f:\mathbf{R}\to\mathbf{R}$ and $\lim_{x\to\infty} f(x) = L$, then $a_n \to L$
(allows us to use L'Hopital's Rule!)

Another basic list: (x = fixed number, k = konstant)

$\frac{1}{n} \to 0 \qquad k \to k \qquad x^{\frac{1}{n}} \to 1$

$n^{\frac{1}{n}} \to 1 \qquad (1 + \frac{x}{n})^n \to e^x \qquad \frac{x^n}{n!} \to 0$

$x^n \to \{$ 0, if $|x| < 1$; 1, if $x = 1$; diverges, otherwise $\}$

Infinite series

An infinite series is an infinite sum of numbers

$$a_1 + a_2 + a_3 + \ldots = \sum_{n=1}^{\infty} a_n \qquad \text{(summation notation)}$$

n-th term of series = a_n ; N-th partial sum of series = $s_N = \sum_{n=1}^{N} a_n$

An infinite series **converges** if the sequence of partial sums $\{s_N\}_{N=1}^{\infty}$ converges

We may start the series anywhere: $\sum_{n=0}^{\infty} a_n$, $\sum_{n=1}^{\infty} a_n$, $\sum_{n=3437}^{\infty} a_n$, etc. ;
convergence is unaffected (but the number it adds up to is!)

Ex. geometric series: $a_n = ar^n$; $\sum_{n=0}^{\infty} a_n = \frac{a}{1-r}$

if $|r| < 1$; otherwise, the series **diverges**.

Ex. Telescoping series: partial sums s_N 'collapse' to a simple expression

E.g. $\sum_{n=1}^{\infty} \frac{1}{n(n+2)} = \sum_{n=1}^{\infty} \frac{1}{2}(\frac{1}{n} - \frac{1}{n+2})$; $s_N = \frac{1}{2}(\frac{1}{1} + \frac{1}{2} - (\frac{1}{N+1} + \frac{1}{N+2}))$

n-th term test: if $\sum_{n=1}^{\infty} a_n$ converges, then $a_n \to 0$

So if the n-th terms **don't** go to 0, then $\sum_{n=1}^{\infty} a_n$ diverges

Basic limit theorems: if $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ converge, then

$\sum_{n=1}^{\infty}(a_n + b_n) = \sum_{n=1}^{\infty} a_n + \sum_{n=1}^{\infty} b_n$ $\quad$ $\sum_{n=1}^{\infty}(a_n - b_n) = \sum_{n=1}^{\infty} a_n - \sum_{n=1}^{\infty} b_n$

$\sum_{n=1}^{\infty}(ka_n) = k\sum_{n=1}^{\infty} a_n$

Truncating a series: $\sum_{n=1}^{\infty} a_n = \sum_{n=N}^{\infty} a_n + \sum_{n=1}^{N-1} a_n$

Comparison tests

Again, think $\sum_{n=1}^{\infty} a_n$, with $a_n \geq 0$ all n

Convergence depends only on partial sums s_N being **bounded**

One way to determine this: **compare** series with one we **know** converges or diverges

Comparison test: If $b_n \geq a_n \geq 0$ for all n (past a certain point), then

if $\sum_{n=1}^{\infty} b_n$ converges, so does $\sum_{n=1}^{\infty} a_n$; if $\sum_{n=1}^{\infty} a_n$ diverges, so does $\sum_{n=1}^{\infty} b_n$

(i.e., smaller than a convergent series converges; bigger than a divergent series diverges)

More refined: Limit comparison test: a_n and $b_n \geq 0$ for all n, $\frac{a_n}{b_n} \to L$

If $L \neq 0$ and $L \neq \infty$, then $\sum a_n$ anf $\sum b_n$ either **both** converge or **both** diverge

If $L = 0$ and $\sum b_n$ converges, then so does $\sum a_n$

If $L = \infty$ and $\sum b_n$ diverges, then so does $\sum a_n$

(Why? eventually $(L/2)b_n \leq a_n \leq (3L/2)b_n$; so can use comparison test.)

Ex: $\sum 1/(n^3 - 1)$ converges; L-comp with $\sum 1/n^3$

$\sum n/3^n$ converges; L-comp with $\sum 1/2^n$

$\sum 1/(n \ln(n^2 + 1)$ diverges; L-comp with $\sum 1/(n \ln n)$

The integral test

Idea: $\sum_{n=1}^{\infty} a_n$ with $a_n \geq 0$ all n, then the partial sums

$\{s_N\}_{N=1}^{\infty}$ forms an increasing sequence;

so converges exactly when bounded from above

If (eventually) $a_n = f(n)$ for a **decreasing** function $f : [a, \infty) \to \mathbf{R}$, then

$$\int_{a+1}^{N+1} f(x)\ dx \leq s_N = \sum_{n=a}^{N} a_n \leq \int_a^N f(x)\ dx$$

so $\sum_{n=a}^{\infty} a_n$ converges exactly when $\int_a^{\infty} f(x)\ dx$ converges

Ex: $\sum_{n=1}^{\infty} \frac{1}{n^p}$ converges exactly when $p > 1$ (p-series)

The ratio and root tests

A series $\sum a_n$ <u>converges</u> <u>absolutely</u> if $\sum |a_n|$ converges.

If $\sum |a_n|$ converges then $\sum a_n$ converges

Previous tests have you compare your series with **something else** (another series, an improper integral); these tests compare a series with itself (sort of)

Ratio Test: $\sum a_n$, $a_n \neq 0$ all n; $\lim_{n\to\infty} |\frac{a_{n+1}}{a_n}| = L$

If $L < 1$ then $\sum a_n$ converges absolutely

If $L > 1$, then $\sum a_n$ diverges

If $L = 1$, then try something else!

Root Test: $\sum a_n$, $\lim_{n\to\infty} |a_n|^{1/n} = L$

If $L < 1$ then $\sum a_n$ converges absolutely

If $L > 1$, then $\sum a_n$ diverges

If $L = 1$, then try something else!

Ex: $\sum \frac{4^n}{n!}$ converges by the ratio test $\quad \sum \frac{n^5}{n^n}$ converges by the root test

Power series

Idea: turn a series into a function, by making the terms a_n <u>depend</u> on x

replace a_n with $a_n x^n$; series of powers

$\sum_{n=0}^{\infty} a_n x^n$ = power series centered at 0

$\sum_{n=0}^{\infty} a_n (x-a)^n$ = power series centered at a

Big question: for what x does it converge? Solution from ratio test

$\lim \left|\frac{a_{n+1}}{a_n}\right| = L$, set $R = \frac{1}{L}$

then $\sum_{n=0}^{\infty} a_n(x-a)^n$ converges absolutely for $|x-a| < R$

diverges for $|x-a| > R$; R = radius of convergence

Ex.: $\sum_{n=0}^{\infty} x^n = \dfrac{1}{1-x}$; conv. for $|x| < 1$

Why care about power series?

Idea: partial sums $\sum_{k=0}^{n} a_k x^k$ are <u>polynomials</u>;

if $f(x)=\sum_{n=0}^{\infty} a_n x^n$, then the poly's make good approximations for f

Differentiation and integration of power series

Idea: if you differentiate or integrate each term of a power series, you get a power series which is the derivative or integral of the original one.

If $f(x) = \sum_{n=0}^{\infty} a_n(x-a)^n$ has radius of conv R,

then so does $g(x) = \sum_{n=1}^{\infty} n a_n (x-a)^{n-1}$, and $g(x) = f'(x)$

<u>and</u> so does $g(x) = \sum_{n=0}^{\infty} \dfrac{a_n}{n+1}(x-a)^{n+1}$, and $g'(x) = f(x)$

Ex: $f(x) = \sum_{n=0}^{\infty} \dfrac{x^n}{n!}$, then $f'(x) = f(x)$, so (since $f(0) = 1$) $f(x) = e^x = \sum_{n=0}^{\infty} \dfrac{x^n}{n!}$

Ex.: $\dfrac{1}{1-x} = \sum_{n=0}^{\infty} x^n$, so $-\ln(1-x) = \sum_{n=0}^{\infty} \dfrac{x^{n+1}}{n+1}$ (for $|x| < 1$), so

(replacing x with $-x$) $\ln(x+1) = \sum_{n=0}^{\infty} \dfrac{(-1)^n x^{n+1}}{n+1}$, so

(replacing x with $x-1$) $\ln(x) = \sum_{n=0}^{\infty} \dfrac{(-1)^n (x-1)^{n+1}}{n+1}$

Ex:. $\arctan x = \int \dfrac{1}{1-(-x^2)}\, dx = \int \sum_{n=0}^{\infty} (-x^2)^n\, dx =$

$$\sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{2n+1} \qquad \text{(for } |x| < 1\text{)}$$

Taylor series

Idea: start with function $f(x)$, find power series <u>for</u> it.

If $f(x) = \sum_{n=0}^{\infty} a_n(x-a)^n$, then (term by term diff.)

$$f^{(n)}(a) = n!a_n \text{ ; So } a_n = \frac{f^{(n)}(a)}{n!}$$

Starting with f, define $P(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n$,

the Taylor series for f, centered at a.

$P_n(x) = \sum_{k=0}^{n} \frac{f^{(k)}(a)}{k!}(x-a)^k$, the n-th Taylor polynomial for f.

Ex.: $f(x) = \sin x$, then $P(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!}x^{2n+1}$

Big questions: Is $f(x) = P(x)$? (I.e., does $f(x) - P_n(x)$ tend to 0 ?)
If so, how well do the P_n's approximate f ? (I.e., how small is $f(x) - P_n(x)$?)

Error estimates

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n$$

means that the value of f at a point x (far from a) can be determined just from the behavior of f near a (i.e., from the derivs. of f at a). This is a very powerful property, one that we wouldn't ordinarily expect to be true. The amazing thing is that it often is:

$$P(x,a) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n \text{ ; } P_n(x,a) = \sum_{k=0}^{n} \frac{f^{(k)}(a)}{k!}(k-a)^n \text{ ;}$$

$R_n(x,a) = f(x) - P_n(x,a) = n$-th remainder term = error in using P_n to approximate f

Taylor's remainder theorem : estimates the size of $R_n(x,a)$

If $f(x)$ and all of its derivatives (up to $n+1$) are continuous on $[a,b]$, then

$$f(b) = P_n(b,a) + \frac{f^{(n+1)}(c)}{(n+1)!}(b-a)^{n+1} \text{ , for some } c \text{ in } [a,b]$$

i.e., for each x, $R_n(x,a) = \frac{f^{(n+1)}(c)}{(n+1)!}(x-a)^{n+1}$, for some c between a and x

so if $|F^{(n+1)}(x)||leqM$ for every x in $[a,b]$, then $|R_n(x,a)| \leq \frac{M}{(n+1)!}(x-a)^{n+1}$ for every x in $[a,b]$

Ex.: $f(x)=\sin x$, then $|f^{(n+1)}(x)| \leq 1$ for all x, so $|R_n(x,0)| \leq \frac{|x|^{n+1}}{(n+1)!} \to 0$ as $n \to \infty$

so $\sin x = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!}x^{2n+1}$

Similarly, $\cos x = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!}x^{2n}$

Use Taylor's remainder to estimate values of functions:

$e^x = \sum_{n=0}^{\infty} \frac{(x)^n}{(n)!}$, so $e{=}e^1{=}\sum_{n=0}^{\infty} \frac{1}{(n)!}$

$|R_n(1,0)| = \frac{f^{(n+1)}(c)}{(n+1)!} = \frac{e^c}{(n+1)!} \le \frac{e^1}{(n+1)!} \le \frac{4}{(n+1)!}$

since $e < 4$ (since $\ln(4) > (1/2)(1) + (1/4)(2) = 1$)

(Riemann sum for integral of $1/x$)

so since $\frac{4}{(13+1)!} = 4.58{\times}10^{-11}$,

$e = 1 + 1 + \frac{1}{2} + \frac{1}{6} + \frac{1}{24} + \frac{1}{120} + \cdots + \frac{1}{13!}$, to 10 decimal places.

Other uses: if you know the Taylor series, it tells you the values of the derivatives at the center.

Ex.: $e^x{=}\sum_{n=0}^{\infty} \frac{(x)^n}{(n)!}$, so

$xe^x = \sum_{n=0}^{\infty} \frac{(x)^{n+1}}{(n)!}$, so

15th deriv of xe^x , at 0, is 15!(coeff of x^{15}) $= \frac{15!}{14!} = 15$

Substitutions: new Taylor series out of old ones

Ex. $\sin^2 x = \frac{1-\cos(2x)}{2} = \frac{1}{2}(1 - \sum_{n=0}^{\infty} \frac{(-1)^n (2x)^{2n}}{(2n)!}$

$= \frac{1}{2}(1 - (1 - \frac{(2x)^2}{2!} + \frac{(2x)^4}{4!} - \frac{(2x)^6}{6!} + \cdots$

$= \frac{2x^2}{2!} - \frac{2^3x^4}{4!} + \frac{2^5x^6}{6!} - \frac{2^7x^8}{8!} + \cdots$

Integrate functions we can't handle any other way:

Ex.: $e^{x^2} = \sum_{n=0}^{\infty} \frac{(x)^2 n}{(n)!}$, so

$\int e^{x^2}\ \mathrm{d}x = \sum_{n=0}^{\infty} \frac{(x)^{2n+1}}{n!(2n+1)}$