Math 1650
Topics for second exam

(Technically, everything covered on the first exam, plus...)

Chapter 2: Polynomials

§3: Polynomial division
- root a of $f \leftrightarrow$ factor $(x-a)$ of $f(x)$
  - reason: polynomial (long) division
- $f(x) = (x-a)g(x) + b$; a=root, then b=0
- polonomial = (divisor)(quotient) + remainder
  - degree of remainder is less than degree of divisor
- synthetic division: fast method to divide by (x-a)

§4: Real zeros of polynomial functions
- $f(x) = a_n x^n + \cdots + a_1 x + a_0$
- 'Counting' zeros of f
  - Descartes' rule of signs
  - p=number of positive roots of f, q=number of negative roots of f
    - (number of changes in sign of coeffs of f) $-$ p is $\geq$0 and even
    - (number of changes in sign of coeffs of $f(-x)$) $-$ q is $\geq$0 and even
- Rational roots test
  - If $a_n, \ldots, a_0$ are all integers, $a_n \neq 0$, and $r = p/q$ is a rational root of f, then q divides a_n evenly and p divides a_0 evenly.
    - backwards: can show roots of a polynomial can't be rational.
- Bounding roots: start with $a_n > 0$.
  - If $c > 0$ and the bottom row after synthetic division of f using c are all $\geq$0,
    - then no root of f is bigger than c.
  - If c <0 and the bottom alternates sign, then no root of f is smaller than c.

§5: Complex numbers
- Some polynomials have no roots, e.g., $f(x) = x^2 + 1$. Invent some!
- $i = \sqrt{-1}$, pretend i behaves like a real number
  - complex numbers: standard form $z = a + bi$; addition, subtraction, multiplication
  - division: complex conjugate $\overline{z} = a - bi$
    - $z \cdot \overline{z} = a^2 + b^2$ (real!) ; $z_1/z_2 = (z_1 \cdot \overline{z_2})/(z_2 \cdot \overline{z_2})$
- $a, b > 0$, then $\sqrt{-a} \cdot \sqrt{-b} = -\sqrt{(-a)(-b)}$ (unfortunately)

§6: The fundamental theorem of algebra
- FTA: Every polynomial $f(x)$ (with coefficients in $\mathbf{C}$ or $\mathbf{R}$) has a
  - complex root r ; $f(r)$=0
  - Every polynomial factors into linear factors (with coefficients in $\mathbf{C}$)
    - FTA says it can be done; it doesn't tell you how to do it!
- Conjugate pairs; if coeffs of f are real, and r is a root, then so is $\overline{r}$
  - $(x-r)(x-\overline{r})$ has real coeffs
  - every polynomial with real coeffs factors in linear and irreducible quadratic factors.

§7: Rational functions
- rational function = quotient of polynomials
- $p(x) = a_n x^n + \cdots + a_0$, $q(x) = b_m x^m + \cdots + b_0$; $f(x) = p(x)/q(x)$
- domain = where $q(x) \neq$0
- vertical asymptote $x = a$: $f(x) \to \pm\infty$ as $x \to a$
- horizontal asymptote: $f(x) \to a$ as $x \to \pm\infty$
  - $n < m$: horiz. asymp. $y = 0$
  - $n = m$: horiz. asymp. $y = a_n/b_m$

$n > m$: no horiz. asymp.

Slant asymptote: $n = m + 1$. Asymp. = linear part from division of $p(x)$ by $q(x)$

Chapter 3: Exponential and logarithmic functions

§1: Exponential functions

exponential expressions a^b

Rules: $a^{b+c} = a^b a^c$; $a^{bc} = (a^b)^c$; $(ab)^c = a^c b^c$

Function $f(x) = a^x$; approximate $f(x)$ by f(rational number close to x)

Domain: $\mathbf{R}$; range: $(0, \infty)$; horiz. asymp. $y = 0$

Graphs:

$a > 1$ $\qquad$ $0 < a < 1$

Most natural base: $e = 2.718281829459045.....$

Exponential growth: compound interest

P=initial amount, r=interest rate, compounded n times/year

$A(t) = P \cdot (1 + r/n)^{nt}$

$n \to \infty$, continuous compounding : $A(t) = Pe^{rt}$

Radioactive decay: half-life = k ($A(k) = A(0)/2$)

$A(t) = A(0)(1/2)^{t/k}$

§2: Logarithmic functions

$\log_a x$ = the number you raise a to to get x

$\log_a x$ is the <u>inverse</u> of a^x

a = base of the logarithm

$\log_a(a^x) = x$, all x ; $a^{\log_a x} = x$, all $x > 0$

Domain: all $x > 0$; range: all x

Graph = reflection of graph of a^x across line $y = x$

vertical asymptote: $x = 0$

natural logarithm: $\log_e x = \ln x$

§3: Properties of logarithms

logarithms undo exponentials; properties are 'reverse' of exponentials

$\log_a(bc) = \log_a b + \log_a c$; $\log_a(b^c) = c \log_a b$

$(\log_b c)(\log_a b) = \log_a(b^{\log_b c}) = \log_a c$; so $\log_b c = \dfrac{\log_a c}{\log_a b}$

E.g., $a = e$: $\log_b c = \dfrac{\ln c}{\ln b}$

§4: Exponential and logarithmic equations

exponential equation: take logs!

$a^{blah} = bleh$, then $(blah)\ln a = \ln(bleh)$

$(2^x - 3)(2^x - 7) = 0$, then 2^x=3 <u>or</u> 2^x=7

logarithmic equation: combine into a single log (one on each side?) and exponentiate both sides

Application: doubling time
time for investment to triple at interest rate of r compounded n times/year:
solve $(1+r/n)^{nt}=3$