Solutions

Name:

Math 221, Section 3

Second Exam 2

Show all work. How you get your answer is just as important, if not more important, than the answer itself. If you think it, write it!

1. Find the general solution to the differential equation

$$x^2y'' + 4xy' + 5y = 0$$

Cauchy-Euler: $r(r-1) + 4r + 5 = 0$ (aux. eqn.)

$$r^2 + 3r + 5 = 0$$

$$r = \frac{-3 \pm \sqrt{9-20}}{2} = -\frac{3}{2} \pm \frac{\sqrt{11}}{2} i$$

Solutions: $y_1 = x^{-3/2}\cos\left(\frac{\sqrt{11}}{2}\ln x\right)$, $y_2 = x^{-3/2}\sin\left(\frac{\sqrt{11}}{2}\ln x\right)$

General solution: $y = c_1 x^{-3/2}\cos\left(\frac{\sqrt{11}}{2}\ln x\right) + c_2 x^{-3/2}\sin\left(\frac{\sqrt{11}}{2}\ln x\right)$.

2. The homogeneous equation

$$y'' - \frac{1}{t}y' + (1 - \frac{1}{t}\tan t)y = 0$$

has, as one solution, the function $y = \cos t$. Use reduction of order to find a second, linearly independent, solution.

Second solution $y = c(t)\cos t$

$p(t) = -1/t$

$\int p(t) = -\ln t$

$$c(t) = \int \frac{e^{-\int p(t)dt}}{(\cos t)^2} dt$$

$e^{-\int p(t)dt} = e^{-(-\ln t)}$

$= e^{\ln t} = t$

$$= \int \frac{t}{\cos^2 t} dt = \int t \sec^2 t \, dt$$

$u = t \quad dv = \sec^2 t \, dt$

$du = dt \quad v = \tan t$

$$= t\tan t - \int \tan t \, dt = t\tan t - \int \frac{\sin t}{\cos t} dt$$

$u = \cos t$

$du = -\sin t \, dt$

$$= t\tan t + \int \frac{du}{u}\Big|_{u=\cos t}$$

$$= t\tan t + \ln u\Big|_{u=\cos t} = t\tan t + \ln(\cos t)$$

So

$$y = c(t)\cdot\cos t = (t\tan t + \ln(\cos t))\cos t$$

$$= t\sin t + \cos t \ln(\cos t)$$

3. Use the method of undetermined coefficients to find a particular solution to the differential equation

$$y'' + y' + 2y = t \sin t$$

$y''+y'+2y=0 \qquad r^2+r+2=0 \qquad r=\frac{-1\pm\sqrt{1-8}}{2}=-\frac{1}{2}\pm\frac{\sqrt{7}}{2}i$

So $\sin t$, $t\sin t$ not homogeneous solutions

Use: $y=(at+b)\sin t+(ct+d)\cos t$

$y'=a\sin t+(at+b)\cos t+c\cos t+(ct+d)(-\sin t)$

$=(-ct+(a-d))\sin t+(at+(b+c))\cos t$

$y''=(-c)\sin t+(-ct+(a-d))\cos t+a\cos t+(at+(b+c))(-\sin t)$

$=(-at-(b+2c))\sin t+(-ct+(2a-d))\cos t$

$y''+y'+2y=\left[(-at-(b+2c))+(-ct+(a-d))+2(at+b)\right]\sin t+$
$\left[(-ct+(2a-d))+(at+(b+c))+2(ct+d)\right]\cos t$

$=(a-c)t\sin t+(a+b-2c-d)\sin t+(a+c)t\cos t$
$+(2a+b+c+d)\cos t$

$=t\sin t$

So $a-c=1$, $a+b-2c-d=0$, $a+c=0$, $2a+b+c+d=0$

So $c=-a$, so $a+a=1$, so $a=\frac{1}{2}$, so $c=-\frac{1}{2}$

$0=a+b-2c-d=b-d+3/2$, so $b-d=-3/2$
$0=2a+b+c+d=b+d+1/2$, so $b+d=-1/2$

So $2b=-2$, so $b=-1$

So $d=-1/2-b=-1/2+1=1/2$

So $\boxed{y=\left(\frac{1}{2}t-1\right)\sin t+\left(-\frac{1}{2}t-\frac{1}{2}\right)\cos t}$

4. The homogeneous differential equation

$$t^2y'' - ty' + y = 0$$

has (fundamental) solutions $y_1 = t$ and $y_2 = t\ln t$ (for $t > 0$). Use variation of parameters to find the **general** solution (for $t > 0$) to the inhomogeneous equation

$$t^2y'' - ty' + y = t^3$$

Standard form: $y'' + \left(\frac{-1}{t}\right)y' + \left(\frac{1}{t^2}\right)y = t$

$y = c_1y_1 + c_2y_2$ where

$$W = \begin{vmatrix} t & t\ln t \\ 1 & 1+\ln t \end{vmatrix} = t(1+\ln t) - (t\ln t)(1) = \underline{\underline{t}}$$

$$c_1' = \frac{\begin{vmatrix} 0 & t\ln t \\ t & 1+\ln t \end{vmatrix}}{W} = \frac{-t^2\ln t}{t} = -t\ln t$$

$$c_2' = \frac{\begin{vmatrix} t & 0 \\ 1 & t \end{vmatrix}}{W} = \frac{t^2}{t} = t$$

So $c_1 = \int \overset{-t^2/2}{-t}\,\underset{1/t}{\ln t}\,dt = -\frac{t^2}{2}\ln t - \int -\frac{t^2}{2}\frac{1}{t}\,dt = -\frac{t^2}{2}\ln t + \int \frac{t}{2}\,dt$

$$= -\frac{t^2}{2}\ln t + \frac{t^2}{4}$$

$$c_2 = \int t\,dt = \frac{t^2}{2}$$

So $y = \left(-\frac{1}{2}t^2\ln t + \frac{1}{4}t^2\right)t + \left(\frac{1}{2}t^2\right)(t\ln t)$

$$= -\frac{1}{2}t^3\ln t + \frac{1}{4}t^3 + \frac{1}{2}t^3\ln t = \boxed{\frac{1}{4}t^3}$$

5. Find the solution to the initial value problem

$$y''' - 3y' - 2y = 0$$

$$y(0) = 0, y'(0) = 0, y''(0) = 1$$

Auxiliary eqn: $r^3 - 3r - 2 = 0$

Possible roots: $r = \pm 1, \pm 2$

Check: $r=1$? $1-3-2=-4$ No.

$r=-1$? $-1+3-2=0$ Yes ✓

$$(r+1)(r^2-r-2)=0$$

$$\hookrightarrow r = \frac{1 \pm \sqrt{1+8}}{2} = \frac{1 \pm 3}{2} \quad \begin{cases} r = \frac{1+3}{2} = 2 \\ r = \frac{1-3}{2} = -1 \end{cases}$$

$r = -1, -1, 2$

General Solutions: $y = c_1 e^{-x} + c_2 x e^{-x} + c_3 e^{2x}$

$$y' = -c_1 e^{-x} + c_2 e^{-x} - c_2 x e^{-x} + 2c_3 e^{2x}$$

$$y'' = c_1 e^{-x} - c_2 e^{-x} - c_2 e^{-x} + c_2 x e^{-x} + 4c_3 e^{2x}$$

$y(0) = c_1 + c_3 = 0 \rightsquigarrow c_3 = -c_1$

$y'(0) = -c_1 + c_2 + 2c_3 = 0 = -3c_1 + c_2 \rightsquigarrow c_2 = 3c_1$

$y''(0) = c_1 - 2c_2 + 4c_3 = c_1 - 6c_1 - 4c_1 = -9c_1 = 1$

$c_1 = -1/9,\ c_2 = -1/3,\ c_3 = 1/9$

Sol $\boxed{y = -\frac{1}{9}e^{-x} - \frac{1}{3}xe^{-x} + \frac{1}{9}e^{2x}}$