Solutions

Math 221 Section 3

Exam 1

Exams provide you, the student, with an opportunity to demonstrate
your understanding of the techniques presented in the course. So:

Show all work. The steps you take to your answer are just as important, if not more important, than the answer itself. If you think it, write it!

1. (20 pts.) Find the (implicit) solution to the initial value problem

$$\frac{dy}{dt} = te^y \qquad y(0) = 5 .$$

Separable: $\frac{dy}{e^y} = t\,dt \qquad \int e^{-y}\,dy = \int t\,dt$

$-e^{-y} = \frac{1}{2}t^2 + C$

$t=0,\ y=5 \qquad -e^{-5} = \frac{1}{2}(0)^2 + C \qquad C = -e^{-5}$

$\boxed{-e^{-y} = \frac{1}{2}t^2 - e^{-5}}$

Explicit solution: $e^{-y} = e^{-5} - \frac{1}{2}t^2$

$-y = \ln\left(e^{-5} - \frac{1}{2}t^2\right)$

$y = -\ln\left(e^{-5} - \frac{1}{2}t^2\right)$

2. (a) (15 pts.) For the autonomous differential equation

$$y' = (y-1)^2(y+1)(y-4)^3$$

find the equilibrium solutions, and determine, for each, whether it is a stable, unstable, or a node.

equilibrium : $y' = 0 = (y-1)^2(y+1)(y-4)^3$

$\Longrightarrow y = 1, \cancel{1}, -1, 4$

y' (+)(−)(−) (+)(+)(−) (+)(+)(−) (+)(+)(+)

++ 0 −− 0 −− 0 ++

>>>>|<<<|<<<|>>>

−1 1 4

$y=-1$ stable

$y=1$ node

$y=4$ unstable

(b) (5 pts.) For a general autonomous equation $y' = f(y)$, if f is **continuous**, what must be true of f to insure that **every** equilibrium solution is *stable*?

need every equilibrium $f(y)$ ++ 0 −− >>|<<

So every root of f has $f>0$ to left and $f<0$ to right

$\Longrightarrow$ at most one root, otherwise $f>0$ to left of one and $f<0$ to right of other $\Longrightarrow$ another root in b/w.

So at most one root of f, and $f>0$ to left of it, $f<0$ to right.

~~there exist~~.

Never mind,

3. (20 pts.) Use Euler's method with a stepsize of $h = 1$ to approximate the solution to the initial value problem

$$y' = ty^2 - t^2 y \qquad y(0) = 2$$

at time $t = 3$.

$t_0 = 0$, $y_0 = 2$ $\quad m_0 = 0 \cdot 2^2 - 0^2 \cdot 2 = 0$

$t_1 = t_0 + h = 0 + 1 = 1$, $y_1 = y_0 + m_0 h = 2 + 0 \cdot 1 = 2$

$m_1 = 1 \cdot 2^2 - 1^2 \cdot 2 = 4 - 2 = 2$

$t_2 = t_1 + h = 1 + 1 = 2$, $y_2 = y_1 + m_1 h = 2 + 2 \cdot 1 = 4$

$m_2 = 2 \cdot 4^2 - 2^2 \cdot 4 = 32 - 16 = 16$

$t_3 = t_2 + h = 2 + 1 = 3$, $y_3 = y_2 + m_2 \cdot h = 4 + 16 \cdot 1 = 20$

So $\boxed{y(3) \approx y_3 = 20}$

4. (20 pts.) Find the general solution to the differential equation

$$\frac{dy}{dx} = (\tan x)y - x$$

$$y' - (\tan x)y = -x$$

$$p(x) = -\tan x \qquad g(x) = -x$$

$$\int p(x) = \int -\tan x\,dx = \int \frac{-\sin x}{\cos x}\,dx = \int \frac{d(\cos x)}{\cos x}$$

$$= \ln(\cos x)$$

$$e^{\int p(x)dx} = e^{\ln(\cos x)} = \cos x$$

$$\int e^{\int p(x)dx} g(x)dx = -\int x \cos x\,dx = -\left(x\sin x - \int \sin x\,d\right.$$

$$= -\left(x\sin x + \cos x + \cancel{c}\right)$$

$$= -x\sin x - \cos x + \cancel{c}$$

$$y = e^{-\int p(x)dx}\left(\int e^{\int p(x)dx} g(x)\,dx + c\right)$$

$$= \frac{1}{\cos x}\left(-x\sin x - \cos x + c\right)$$

$$\boxed{= -x\tan x - 1 + c\sec x}$$

<u>Check!</u> $y' = -\tan x - x\sec^2 x + c\sec x\tan x$

$$(\tan x)y - x = -x\tan^2 x - \tan x + c\sec x\tan x - x$$

$$= -x(\sec^2 x - 1) - \tan x + c\sec x\tan x - x$$

$$= -x\sec^2 x - \tan x + c\sec x\tan x$$

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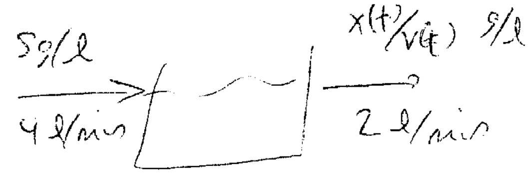
5. (20 pts.) A 300 liter vat initially contains 200 liters of a salt solution with a concentration of 3 grams/liter. Solution with a concentration of 5 grams/liter is flowing into the vat at a rate of 4 liters/minute, while the (well-stirred) vat is draining off at a rate of 2 liters/minute. What will the concetration of solution in the vat be when the vat starts to overflow?



$$V(0) = 200$$

$$x(0) = 3 \cdot 200 = 600\ g$$

$$V(t) = 200 + (4-2)t = 200 + 2t$$

$$x' = 5 \cdot 4 - \frac{x}{V} \cdot 2 = 20 - \frac{1}{100+t} x$$

what is $\frac{x(t)}{V(t)}$ when $V(t) = 200 + 2t = 300$? $\boxed{t = 50\ ?}$

$$x' + \frac{1}{t+100} x = 20 \qquad \int \frac{1}{t+100}\,dt = \ln(t+100)$$

$$x(t) = e^{-\int^t p(x)dx}\left(\int e^{\int^t p(x)dx} g(t)\,dt + c\right)$$

$$= (t+100)^{-1}\left(\int 20(t+100)\,dt + c\right)$$

$$= (t+100)^{-1}\left(10(t+100)^2 + c\right) = 10(t+100) + c(t+100)^{-1}$$

$$x(0) = 600 = 1000 + c(100)^{-1} \qquad -400 = c(100)^{-1}$$

$$c = -40{,}000$$

$$x(t) = 10(t+100) - 40{,}000(t+100)^{-1}$$

$$x(50) = 1500 - 40000/150$$

concentration $= \frac{x(50)}{V(50)} = \frac{1500 - \frac{40000}{150}}{300} = 5 - \frac{40000}{45000}$

$$= 5 - \frac{8}{9} = \boxed{\frac{37}{9}}\ g/l$$