Solutions

Name:

Math 221, Section ~~0~~3

Exam 2

Show all work. How you get your answer is just as important, if not more important, than the answer itself. If you think it, write it!

1. Find the solution to the initial value problem

$$x^2y'' - 6xy' + 10y = 0$$
$$y(1) = 3 \quad , \quad y'(1) = 5$$

Cauchy-Euler : Try $y = x^r$, $y' = rx^{r-1}$, $y'' = r(r-1)x^{r-2}$

$x^2y'' - 6xy' + 10y = (r(r-1) - 6r + 10)x^r = 0$

$\Rightarrow r^2 - r - 6r + 10 = r^2 - 7r + 10 = 0$

$r = \dfrac{7 \pm \sqrt{49-40}}{2} = \dfrac{7 \pm \sqrt{9}}{2} = \dfrac{7 \pm 3}{2}$

$r = \frac{7-3}{2} = \frac{4}{2} = 2$, $r = \frac{7+3}{2} = \frac{10}{2} = 5$

$y = c_1 x^2 + c_2 x^5$

$y' = 2c_1 x + 5c_2 x^4$

$y(1) = 3 = c_1 + c_2$

$y'(1) = 5 = 2c_1 + 5c_2$

$-1 = 5 - 2\cdot 3 = 3c_2 \Rightarrow c_2 = -1/3 \Rightarrow c_1 = 3 - c_2 = 3 + \frac{1}{3}$
$= 10/3$

$$\boxed{y = \frac{10}{3}x^2 - \frac{1}{3}x^5}$$

2. The homogeneous equation

$$y'' + \frac{1}{t}y' + \frac{2t^2+1}{t^2(t^2+1)^2}y = 0$$

has, as one solution, the function $y_1 = \dfrac{(t^2+1)^{1/2}}{t}$. Use reduction of order to find a second, linearly independent, solution.

$$y_2 = c(t)\,y_1 \quad \underline{\text{where}} \quad c(t) = \int \frac{e^{-\int p(t)\,dt}}{y_1^2}\,dt$$

$$p(t) = \frac{1}{t} \implies \int p(t)\,dt = \ln t, \quad e^{-\int p(t)\,dt} = e^{-\ln t} = e^{\ln t^{-1}} = t^{-1}$$

$$c(t) = \int \frac{t^{-1}\,dt}{\left(\frac{(t^2+1)^{1/2}}{t}\right)^2} = \int \frac{t^2\,t^{-1}}{t^2+1}\,dt = \int \frac{t\,dt}{t^2+1}$$

$$\boxed{u = t^2+1 \quad du = 2t\,dt}$$

$$= \frac{1}{2}\int \frac{du}{u}\Big|_{u=t^2+1} = \frac{1}{2}\ln u\Big|_{u=t^2+1}$$

$$= \frac{1}{2}\ln(t^2+1)$$

$$\boxed{y_2 = c(t)\,y_1(t) = \left(\frac{1}{2}\ln(t^2+1)\right)\frac{(t^2+1)^{1/2}}{t}}$$

3. Use the method of undetermined coefficients to find ~~the general~~ a particular solution to the differential equation

$$y'' + y' - 3y = t^2e^{-2t}$$

t^2e^{-2t} solves homogeneous eqn?

$r^2+r-3=0 \qquad r=\frac{-1\pm\sqrt{1+12}}{2}=\frac{-1\pm\sqrt{13}}{2}$ No.

Use $y=(at^2+bt+c)e^{-2t}$

$y'=(2at+b)e^{-2t}+(-2)(at^2+bt+c)e^{-2t}$

$=(-2at^2+(2a-2b)t+(b-2c))e^{-2t}$

$y''=(-4at+(2a-2b))e^{-2t}+(-2)(-2at^2+(2a-2b)t+(b-2c))e^{-2t}$

$=(4at^2+(4b-8a)t+(2a-4b+4c))e^{-2t}$

$y''+y'-3y=\Big(\big(+4a-2a-3a\big)t^2+\big(4b-8a+2a-2b-3b\big)t$

$+\big(2a-4b+4c+b-2c-3c\big)\Big)e^{-2t}$

$=(1\cdot t^2+0\cdot t+0)e^{-2t}$

$\Rightarrow 4a-2a-3a=-a=1, \quad 4b-8a+2a-2b-3b=-6a-b=0$

$2a-4b+4c+b-2c-3c=2a-3b-c=0$

$\Rightarrow a=-1, \quad b=-6a=6, \quad c=2a-3b=-2-18=-20$

$\boxed{y=(-t^2+6t-20)e^{-2t}}$ particular solution

General solution: $\boxed{c_1e^{\frac{-1+\sqrt{13}}{2}t}+c_2e^{\frac{-1-\sqrt{13}}{2}t}+(-t^2+6t-20)e^{-2t}}$

4. The homogeneous differential equation

$$y'' - \frac{2}{t}y' + \frac{2}{t^2}y = 0$$

has (fundamental) solutions $y_1 = t$ and $y_2 = t^2$ (for $t > 0$). Use variation of parameters to find ~~a particular~~ the general solution (for $t > 0$) to the inhomogeneous equation

$$y'' - \frac{2}{t}y' + \frac{2}{t^2}y = t\sin t = g(t)$$

$y_1 = t \quad y_2 = t^2$

$y_1' = 1 \quad y_2' = 2t$

$$W = \begin{vmatrix} t & t^2 \\ 1 & 2t \end{vmatrix} = 2t^2 - t^2 = t^2$$

$$y = c_1 y_1 + c_2 y_2$$

$$c_1 = \int \frac{\begin{vmatrix} 0 & t^2 \\ g & 2t \end{vmatrix}}{t^2}\,dt = \int \frac{-t^2(t\sin t)}{t^2}\,dt = -\int \underset{\underset{1}{\downarrow}}{t}\,\overset{\overset{-\cos t}{\uparrow}}{\sin t}\,dt$$

$$= -\left[-t\cos t + \int \cos t\,dt\right] = -\left[-t\cos t + \sin t\right]$$

$$= t\cos t - \sin t$$

$$c_2 = \int \frac{\begin{vmatrix} t & 0 \\ 1 & g \end{vmatrix}}{t^2}\,dt = \int \frac{t(t\sin t)}{t^2}\,dt = \int \sin t\,dt \quad -\cos t$$

$$y = (t\cos t - \sin t)t + (-\cos t)t^2$$

$$= t^2\cos t - t\sin t - t^2\cos t = \boxed{-t\sin t}$$

$$\boxed{\text{General solution: } y = c_1 t + c_2 t^2 - t\sin t}$$

5. One solution to the homogeneous equation

$$y'''' + y''' - 7y'' - 13y' - 6y = 0$$

is $y = e^{-x}$. Find the general solution to the homogeneous equation.

Auxiliary equation: $r^4+r^3-7r^2-13r-6=0$

$y=e^{-x}$ is a solution $\Rightarrow$ $r=-1$ is a root.

So $r^4+r^3-7r^2-13r-6=(r+1)(r^3-7r-6)$

roots of $r^3-7r-6=0$?

Possible roots: $r=1,-1,2,-2,+3,-3,6,-6$

$r=1$? $1-7-6=-12 \neq 0$ no.

$r=-1$? $-1+7-6=0$ ✓ yes.

$r^3-7r-6=(r+1)(r^2-r-6)$

$r^2-r-6=(r-3)(r+2)=0$ $r=+3,-2$

Roots are $-1,-1,-2,3$.

$\Rightarrow$ Solutions are $e^{-x}, xe^{-x}, e^{-2x}, e^{3x}$

$$\boxed{\text{General solution: } y=c_1e^{-x}+c_2xe^{-x}+c_3e^{-2x}+c_4e^{3x}}$$