Math 221

Topics for the second exam

Don't forget the topics for the first exam!

Basic object of study: second order linear differential equations

$$y'' + p(t)y' + q(t)y = g(t) \qquad (*)$$

Initial value problem:

$$y(t_0)=y_0 \qquad \text{and} \qquad y'(t_0)=y'_0$$

Basic fact: if $p(t)$, $q(t)$, and $g(t)$ are continuous on an interval around t_0, then any initial value problem has a *unique* solution on that interval. Our Basic goal: find the solution!
Homogeneous: $g(t) = 0$ Constant coefficients: $p(t)$ and $q(t)$ are constant.

Operator notation: write $L[y] = y'' + p(t)y' + q(t)y$ (this is called a *linear operator*), then a solution to (*) is a function y with $L[y] = g(t)$.
For a linear differential equation, $L[c_1y_1 + c_2y_2] = c_1L[y_1] + c_2L[yt]$, and so if y_1 and y_2 are both solutions to $L[y] = 0$ then so is $c_1y_1 + c_2y_2$. $c_1y_1 + c_2y_2$ is called a *linear combination* of y_1 and y_2. This is called the *Principle of Superposition*: more generally, if $L[y_1] = g_1(t)$ and $L[y_2] = g_2(t)$, then $L[y_1 + y_2] = g_1(t) + g_2(t)$.

With (the right) *two* solutions y_1, y_2 to a *homogeneous* equation

$$y'' + p(t)y' + q(t)y = 0 \qquad (**)$$

we can solve *any* initial value problem, by choosing the right linear combination: we need to solve

$$\begin{aligned} c_1y_1(t_0) + c_2y_2(t_0) &= y_0 \\ c_1y'_1(t_0) + c_2y'_2(t_0) &= y'_0 \end{aligned}$$

for the constants c_1 amd c_2; then $y = c_1y_1 + c_2y_2$ is our solution. This we can do directly, as a pair of linear equations, by solving one equation for one of the constants, and plugging into the other equation, or we can use the formulas

$$c_1 = \frac{\begin{vmatrix} y_0 & y_2(t_0) \\ y'_0 & y'_2(t_0) \end{vmatrix}}{\begin{vmatrix} y_1(t_0) & y_2(t_0) \\ y'_1(t_0) & y'_2(t_0) \end{vmatrix}} \qquad c_2 = \frac{\begin{vmatrix} y_1(t_0) & y_0 \\ y'_1(t_0) & y'_0 \end{vmatrix}}{\begin{vmatrix} y_1(t_0) & y_2(t_0) \\ y'_1(t_0) & y'_2(t_0) \end{vmatrix}}$$

where $\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$. This makes it clear that a solution *exists* (i.e., we have the 'right' pair of functions), provided that the quantity

$$W = W(y_1, y_2)(t_0) = \begin{vmatrix} y_1(t_0) & y_2(t_0) \\ y'_1(t_0) & y'_2(t_0) \end{vmatrix} \neq 0$$

W is called the *Wronskian* (determinant) of y_1 and y_2 at t_0 . The Wronskian is closely related to the concept of *linear independence* of a collection $y_1, \ldots, y_n$ of functions; such a collection is linearly independent if the only linear combination $c_1y_1 + \cdots + c_ny_n$ which is equal to the 0 function is the one with $c_1 = \cdots = c_n = 0$.

Two functions y_1 and y_2 are linearly independent if their Wronksian is non-zero at *some* point; for a pair of solutions to (**), it turns out that the Wronskian is always equal to a constant multiple of

$$exp(-\int p(t)\ dt)$$

and so is either *always* 0 or *never* 0. We call a pair of linearly independent solutions to (**) a pair of *fundamental solutions*. By our above discussion, we can solve any initial value problem for (**) as a linear combination of fundamental solutions y_1 and y_2. By our existence and uniqueness result, this give us:

If y_1 and y_2 are a fundamental set of solutions to the differential equation (**), then *any* solution to (**) can be expressed as a linear combination $c_1y_1 + c_2y_2$ of y_1 and y_2.

So to solve an initial value problem for (**), all we need is a pair of fundamental solutions.

Homogenous equations with constant coefficients: $ay'' + by' + cy = 0$

Basic idea: *guess* that $y = e^{rt}$, and plug in! Get:

$$(ar^2 + br + c)e^{rt} = 0 \quad \text{, so} \quad ar^2 + br + c = 0$$

Solve: get (typically) two roots r_1, r_2, so $y_1 = e^{r_1t}$ and $y_2 = e^{r_2t}$ are *both* solutions.
The equation $ar^2 + br + c = 0$ is called the *auxiliary equation* for our differential equation.

If the roots of the characteristic equation are real and distinct, $r_1 \neq r_2$, then a fundamental set of solutions is

$$y_1 = e^{r_1t}, y_2 = e^{r_2t}$$

If the roots of the characteristic equation are complex $\alpha \pm \beta i$, then a fundamental set of solutions is

$$y_1 = e^{\alpha t}\cos(\beta t), y_2 = e^{\alpha t}\sin(\beta t)$$

If the roots of the characteristic equation are *repeated* (and therefore real), $r_1 = r_2 = r$, then a fundamental set of solutions is

$$y_1 = e^{rt}, y_2 = te^{rt}$$

Reduction of order is a general technique for finding a second, linearly independent, solution y_2 to (**), given a (non-zero) solution y_1; if y_1 is a solution to (**), then so is

$$y_2(t) = y_1(t)\int \frac{\exp(-\int p(t)\ dt)}{(y_1(t))^2}\ dt$$

This formula was found by *assuming* that $y_2(t) = c(t)y_1(t)$, and then determining what differential equation $c(t)$ must satisfy! It turns out to be a first-order equation (hence the name reduction of order).

Higher order equations: Much of what we just did for second order equations goes through without any change for even higher order (linear) equations:

$$L[y] = y^{(n)} \ = a_1(t)y^{(n-1)} + \cdots + a_{n-1}(t)y' + a_n(t)y = g(t) \qquad (!)$$

and its associated homogeneous equation

$$y^{(n)} \ = a_1(t)y^{(n-1)} + \cdots + a_{n-1}(t)y' + a_n(t)y = 0 \qquad (!!)$$

In this case the correct notion of an initial value problem requires us to specify the values, at t_0, of y and all its derivatives up to the $(n-1)$st:

$$y(t_0) = y_0, y'(t_0) = y'_0, \dots, y^{(n-1)}(t_0) = y_0^{(n-1)}$$

As with the second order case, we have a principle of superposition: $L[y_1] = g_1$ and $L[y_2] = g_2$, then $L[y_1 + y_2] = g_1 + g_2$. This means that linear combinations of solutions to the homogeneous equation (!!) are also solutions. And the general solution to (!!) can always be obtained (uniquely) as a linear combination of n *linearly independent* (or *fundamental*) solutions. Linear independence can be determined by computing a Wronskian determinant $W(y_1, \dots, y_n)$ (which we will not develop here).

The theory we developed for homogeneous equations with constant coefficients can be similarly extended. The equation

$$a_0 y^{(n)} + a_1 y^{(n-1)} + \cdots + a_{n-1} y' + a_n y = 0$$

has a fundamental set of solutions determined by its auxiliary equation

$$a_0 r^n + \cdots + a_{n-1} r + a_n = 0$$

Real roots r correspond to solutions $\exp(rt)$; complex roots to solutions $\exp(\alpha t)\cos(\beta t)$ and $\exp(\alpha t)\sin(\beta t)$. The only extra wrinkle is that we can have repeated roots which repeat *many* times, and even repeated complex roots! For each, we do as we did before and create new fundamental solutions by multiplying our basic solution by t, as many times as the root repeats. For example, the equation

$$y^{(4)} + 2y'' + y = 0$$

has a characteristic equation with roots $i, i, -i$, and $-i$, and so its fundamental solutions are

$$\cos(t), t\cos(t), \sin(t), \text{ and } t\sin(t)$$

Inhomogeneous linear equations: We can solve an inhomogeneous equation

$$L[y] = y^{(n)} \; + a_1(t) y^{(n-1)} + \cdots + a_{n-1}(t) y' + a_n(t) y = g(t) \qquad (!)$$

with $g(t) \neq 0$, by using our knowledge of the solution to its associated homogeneous equation. The principle of superposition tells us that for any pair of solutions Y_1, Y_2 to (!), $L[Y_2 - Y_1] = 0$, and so if we have a fundamental set of solutions to the associated homogeneous equation, $y_1, \dots, y_n$, we can write

$$Y_2 = Y_1 + c_1 y_1 + \cdots + c_n y_n$$

In other words, we can find *any* solution to (!) by finding one *particular* solution, together with a fundamental set of solutions to the associated homogeneous equation (!!). Any initial value problem can then be solved by solving the system of equations

$$Y_1(t_0) + c_1 y_1(t_0) + \cdots + c_n y_n(t_0) = g(t_0)$$
$$Y'_1(t_0) + c_1 y'_1(t_0) + \cdots + c_n y'_n(t_0) = g'(t_0)$$

all the way to

$$Y_1^{(n-1)}(t_0) + c_1 y_1^{(n-1)}(t_0) + \cdots + c_n y_n^{(n-1)}(t_0) = g^{(n-1)}(t_0)$$

for the constants $c_1, \dots, c_n$.

The only part of this we haven't really explored yet is finding a particular solution to (!). for this we have two techniques.

Variation of parameters: the idea is to start with a pair of fundamental solutions y_1, y_2 to the associated homogeneous equation

$$y'' + p(t)y' + q(t)y = 0 \qquad (**)$$

and then *guess* that the solution to our inhomogeneous equation

$$y'' + p(t)y' + q(t)y = g(t) \qquad (*)$$

is of the form $y(t) = c_1(t)y_1(t) + c_2(t)y_2(t)$, and plug in. The resulting equation is too complicated, but if we make the simplifying assumption

$$c_1'(t)y_1(t) + c_2'(t)y_2(t) = 0$$

then the equation becomes

$$c_1'(t)y_1'(t) + c_2'(t)y_2'(t) = g(t)$$

which we can solve:

$$c_1' = \frac{\begin{vmatrix} 0 & y_2(t_0) \\ g & y_2'(t_0) \end{vmatrix}}{\begin{vmatrix} y_1(t_0) & y_2(t_0) \\ y_1'(t_0) & y_2'(t_0) \end{vmatrix}} \qquad c_2' = \frac{\begin{vmatrix} y_1(t_0) & 0 \\ y_1'(t_0) & g \end{vmatrix}}{\begin{vmatrix} y_1(t_0) & y_2(t_0) \\ y_1'(t_0) & y_2'(t_0) \end{vmatrix}}$$

Here again, the by now familiar Wronskian appears! Note that we must still *integrate* these functions, to determine c_1 and c_2 .

Method of Undetermined Coefficients: Our second approach to solving inhomogeneous equation involves "educated guessing".

Important: This generally works only for equations with constant coefficients!

The basic idea behind the technique is that for most kinds of functions, like polynomials, expoential, sines and cosines, or products of these, all of the derivatives of the function are of the *same basic form.* So if the function $g(t)$ in

$$L[y] = a_0 y^{(n)} + a_1 y^{(n-1)} + \cdots + a_{n-1}y' + a_n y = g(t) \qquad (!)$$

is one of these kinds, what we do is *guess* that our solution y is the same kind. We include *undetermined coefficients*in the solution (hence the name), and by plugging into the differential equation and setting equal to our target function, we *solve* for the undetermined coefficents. In particular,

If g is a polynomial of degree n, we set y to be a (different) polynomial of degree n,

If g is a multiple of an exponential $\exp(rt)$, we set y to be a multiple $c\exp(rt)$ of g,

If g is a multiple of $\sin(\beta t)$ or $\cos(\beta t)$, we set y to be a linear combination $a\sin(\beta t) + b\cos(\beta t)$,

If $g(t) = \exp(rt)\cos(\beta t)$ (or $\exp(rt)\sin(\beta t)$), we set y to be $a\exp(rt)\sin(\beta t) + b\exp(rt)\cos(\beta t)$,

If g is a polynomial of degree n times one of these, we set y to be a (different, unknown) polynomial of degree n times the corresponding function above.

Then we must plug this function into (!), and solve for the undetermined coefficients.

Of course, there is one wrinkle; sometimes our choice of y *cannot* work, because it is a solution to the associated *homogeneous* equation. For example, for the equation

$$L[y] = y'' + y = \cos(t)$$

the function $y = a\cos(t) + b\sin(t)$ will never solve it, because for such a function, $L[y] = 0$. In this case what we must do is multiply our guess by t, or more generally, by a lowest power of t to insure that our guess is not a solution to the homogeneous equation. For this, we must first determine the number of times the root which corresponds to our target solution occurs among the roots of the associated characteristic equation. This can be a trifle tricky to determine; for example, for the equation

$$y'' - 2y' + y = te^t$$

we should guess that our solution is $y = t(at + b)e^t$, since our original guess would be $y = (at + b)e^t$, but this is a solution to the homogeneous equation, while t times it is not; but for

$$y'' - 2y' + y = 3e^t$$

we should guess that our solution is $y = at^2e^t$, since te^t is still a solution to the homogeneous equation, but $y = t^2e^t$ is not.

Finally, if our function $g(t)$ is a linear combination of such functions, we can use this method to solve $L[y] =$ each piece, and then use the Principle of Superposition to find our solution by taking a linear combination.

Higher order equations: The method of undetermined coefficients works equally well for higher order inhomogeneous equations with constant coefficients; the exact same steps will lead you to a solution.

Applications: spring - mass problems

Basic setup: an object with mass m sits on a track and is attached to an immovable wall by a spring. At rest, the mass sits at a point in the track which we will call 0. The mass is then displaced from this equilibrium position and released (with some initial velocity). The position at time t of the object is $x(t)$.
Newton: $mu'' =$ sum of the forces acting on the object. These include:

the spring: $F_s = -kx$ (Hooke's Law; $k > 0$)
friction: $F_f = -bx'$ ($b \geq 0$
a possible external force: $F_e = f(t)$

Putting them all together, we get $mx'' = -kx - bx' + f(t)$, i.e.,

$$mx'' + bx' + kx = f(t)$$

and this is an equation we know how to solve!

Some special cases:

No friction ($b = 0$) , i.e., *undamped*; no external force, i.e., *unforced*). Solutions are

$$u = c_1\cos(\omega_0 t) + c_2\sin(\omega_0 t) = C\cos(\omega_0 t - \delta)$$

where $\omega_0 = \sqrt{k/m}$ = the *natural frequency* of the system, C = amplitude of the vibration, δ (= 'delay') = phase angle, *where*

$C = \sqrt{(c_1^2 + c_2^2)}$, $\tan(\delta) = c_2/c_1$

[[you are not responsible for these formulas; they are included FYI only.]]

$T = 2\pi/\omega_0$ = period of the vibration. Note that a stiffer spring (= larger k) gives higher frequency, shorter period. Larger m gives the opposite.

Damped unforced vibrations: solve the auxiliary equation, solutions have $b^2 - 4km =$ discriminant inside of the square root, and so the solutions depend on the *sign* of the discriminant.

$b^2 > 4km$ (overdamped); fundamental solutions are $e^{r_1 t}, e^{r_2 t}$, $r_1, r_2 < 0$ (roots are negative because $m, b, k > 0$)

$b^2 = 4km$ (critically damped); fundamental solutions are e^{rt}, te^{rt}, $r < 0$

$b^2 < 4km$ (underdamped); fundamental solutions are $e^{rt}\cos(\omega t), e^{rt}\sin(\omega t)$, $r < 0$, $\omega = \sqrt{\omega_0^2 - (b/2m)^2}$

In each case, solutions tend to 0 as t goes to ∞. In first two cases, the solution has at most one local max or min; in the third, it continues to oscillate forever.

Forced vibrations: Focus on periodic forcing term: $f(t) = F_0 \cos(\omega t)$.

Damped case: if we include friction ($b \neq 0$), then the solution turns out to be

$$x = \text{homog. soln. } + C\sin(\omega t - \delta)$$

But since $b > 0$, the homogeneous solutions will tend to 0 as $t \to \infty$; they are called the *transient solution.* (Basically, they just allow us to solve any initial value problem. We can then conclude that any energy given to the system is dissipated over time; leaving only the energy imparted by the forcing term to drive the system along.) The other term is called the *forced response*, or *steady-state solution.*

Undamped: when $\omega = \omega_0$, our forcing term is a solution to the homogeneous equation, so the general solution, instead, is

$$x = C_1 \sin(\omega_0 t - \delta_1) + C_2 t \sin(\omega_0 t - \delta_2)$$

In this case, as t goes to ∞, the amplitude of the second oscillation goes to ∞; the solution, essentially, resonates with the forcing term. (Basically, you are 'feeding' the system at it's natural frequency.) This illustrates the phenomenon of *resonance.*

[[**And a little extra, FYI...**]]

If $\omega \neq \omega_0$, then (using undetermined coefficients) the solution is

$$x = C\cos(\omega_0 t - \delta) + \frac{F_0}{m(\omega_0^2 - \omega^2)}\cos(\omega t)$$

This is the sum of two vibrations with different frequencies.

In the special case $x(0) = 0, x'(0) = 0$ (starting at rest), we can further simplify:

$$x = \frac{2F_0}{m(\omega_0^2 - \omega^2)} \sin(\frac{\omega_0 - \omega}{2}t) \sin(\frac{\omega_0 + \omega}{2}t)$$

When ω is close to ω_0, this illustrates the concept of *beats*; we have a high frequency vibration (the second sine) with *amplitude* a low frequency vibration (the first sine). the mass essentially vibrates rapidly between to sine curves.