Solutions

Name:

Math 221 Section 3

Exam 1

Exams provide you, the student, with an opportunity to demonstrate your understanding of the techniques presented in the course. So:

Show all work. The steps you take to your answer are just as important, if not more important, than the answer itself. If you think it, write it!

1. (20 pts.) Find the (implicit) solution to the initial value problem

$$\frac{dy}{dt} = \frac{te^{y+t}}{y} \qquad y(0) = 2 \ .$$

$$\frac{dy}{dt} = \left(\frac{e^y}{y}\right)\left(te^t\right)$$

$$\int ye^{-y}\,dy = \int te^t\,dt$$

$u=y \quad dv=e^{-y}dy$ $\qquad$ $u=t \quad dv=e^t dt$

$du=dy \quad v=-e^{-y}$ $\qquad$ $du=dt \quad v=e^t$

$$-ye^{-y}+\int e^{-y}dy = te^t - \int e^t dt$$

$$-ye^{-y}-e^{-y} = te^t - e^t + C$$

$t=0, y=2$: $\quad -2e^{-2}-e^{-2} = 0\cdot 1 - 1 + C$

$$C = 1-2e^{-2}-e^{-2} = 1-3e^{-2}$$

$$\boxed{-ye^{-y}-e^{-y} = te^t - e^t + 1 - 3e^{-2}}$$

$$(y+1)e^{-y} = (1-t)e^t + 3e^{-2} - 1$$

2. (15 pts.) Use Euler's method with a stepsize of $h = 1$ to approximate the solution to the initial value problem

$$y' = ty^3 - t^3y \qquad y(0) = 1$$

at time $t = 3$.

How would we alter this problem to find a better approximation to $y(3)$?

$t_0 = 0 \quad y_0 = 1 \qquad m = 0\cdot 1 - 0 \cdot 1 = 0$

$t_1 = 1 \quad y_1 = 1 + 0 \cdot 1 = 1 \qquad m = 1 \cdot 1 - 1 \cdot 1 = 0$

$t_2 = 2 \quad y_2 = 1 + 0 \cdot 1 = 1 \qquad m = 2 \cdot 1 - 8 \cdot 1 = -6$

$t_3 = 3 \quad y_3 = 1 + (-6) \cdot 1 = -5$

$\boxed{y(3) \approx -5}$

To get a better approximation, use a smaller stepsize!

E.g. $h = \frac{1}{2}$ or (better) $h = \frac{1}{10}$ or (better!) $h = \frac{1}{100}$ ....

3. (20 pts.) Find the general solution to the differential equation

$$\frac{dy}{dx} = e^{x^2} - \frac{1}{x} y$$

for $x > 0$.

$y' + \frac{1}{x} y = e^{x^2}$ $\quad p(x) = \frac{1}{x} \quad g(x) = e^{x^2}$

$\int p(x) = \int \frac{1}{x} dx = \ln|x| = \ln x$ (since $x > 0$)

$e^{\int p(x)dx} = e^{\ln x} = x$

$\int g(x) e^{\int p(x)dx} dx = \int x e^{x^2} dx$ $\quad u = x^2$, $du = 2x\,dx$, $x\,dx = \frac{1}{2} du$

$= \frac{1}{2} \int e^u du \Big|_{u=x^2} = \frac{1}{2} e^u \Big|_{u=x^2} = \frac{1}{2} e^{x^2}$

$y = e^{-\int p(x)dx} \left(\int g(x) e^{\int p(x)dx} dx + c \right)$

$= e^{-\ln x} \left(\frac{1}{2} e^{x^2} + c \right) = \frac{1}{x} \left(\frac{1}{2} e^{x^2} + c \right)$

$= \boxed{\frac{1}{2x} e^{x^2} + \frac{c}{x}}$

4. (20 pts.) A vat initially contains 300 liters of a salt solution with a concentration of 4 grams/liter. Solution with a concentration of 5 grams/liter is flowing into the vat at a rate of 2 liters/minute, while the (well-stirred) contents of the vat are draining off at a rate of 3 liters/minute. What will the concetration of solution in the vat be when the vat contains 200 liters of solution?

$V(0) = 300\,l$ initial concentration $= 4\,g/l$

initial amount $= A(0) = 4 \cdot 300 = 1200\,g$

in: $5 \cdot 2$ g/min out: $\frac{A(t)}{V(t)} \cdot 3$ g/min

$$A' = 10 - \frac{A}{V(t)} \cdot 3$$

$$V(t) = 300 + (2-3)t = 300 - t$$

$$A' = 10 - \frac{3}{300-t} A$$

$$V(t) = 200 = 300 - t \qquad t = 100$$

what is $\frac{A(100)}{V(100)}$?

$$A' + \left(\frac{3}{300-t}\right) A = 10 \qquad p(t) = \frac{3}{300-t} \quad g(t) = 10$$

$$\int \frac{3\,dt}{300-t} \quad \begin{pmatrix} u = 300-t \\ du = -dt \\ dt = -du \end{pmatrix} = -3 \int \frac{du}{u} \Big|_{u=300-t} = -3 \ln|u| \Big|_{u=300-t}$$

$$= -3 \ln(300-t)$$

$$e^{\int p(t)\,dt} = e^{-3\ln(300-t)} = (300-t)^{-3}$$

$$\int g(t) e^{\int p(t)\,dt} = \int 10 (300-t)^{-3}\,dt = \frac{10}{(-2)} (300-t)^{-2}(-1)$$

$$= 5(300-t)^{-2} \qquad A(t) = (300-t)^3 \left(5(300-t)^{-2} + C\right)$$

$$= \boxed{5(300-t) + C(300-t)^3}$$

$$A(0) = 1200 = 1500 + C(300)^3 \qquad -300 = C(300)^3 \qquad C = \frac{-1}{(300)^2}$$

$$A(t) = 5(300-t) - \frac{(300-t)^3}{(300)^2}$$

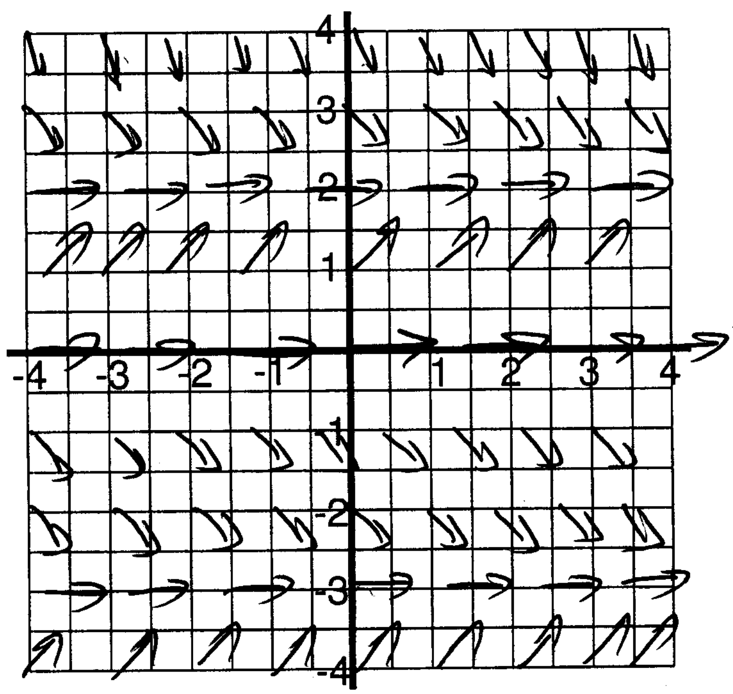
$$\frac{A(100)}{V(100)} = \boxed{\frac{5(200) - \frac{(200)^3}{(300)^2}}{(200)}} = 5 - \left(\frac{2}{3}\right)^2$$

4

5. (25 pts.) A new model of population growth predicts that

$$\frac{dp}{dt} = kp - (a + bp)p^2$$

(i.e., rising population will lead to ever increasing death rates due to "unnatural causes"). Observation suggests that the correct constants for the equation are $k = 6, a = 1, b = 1$. Sketch the direction field for this equation (hint: paying attention to the horizontal tangents will help), and find the (implicit) solutions to the resulting differential equation. Describe the behavior of the solution with $p(0) = 3$, for large t; what limit, if any, does $p(t)$ tend to?.



$p' = 6p - (1+p)p^2$

$= p(6 - p - p^2)$

$= p(3+p)(2-p)$

$p' = 0$: (horizontal tangents)

$p = 0$, $3 + p = 0$, $2 - p = 0$

$p = 0$, $p = -3$, $p = 2$

$p = 1$: $p' = (1)(4)(1) = 4 > 0$

$p = 3$: $p' = (3)(6)(-1) = -18 < 0$

$p = -1$: $p' = (-1)(2)(3) = -6 < 0$

$p = -4$: $p' = (-4)(-1)(6) = 24 > 0$

$p = -2$: $p' = (-2)(1)(4) = -8 < 0$

$p = 4$: $p' = (4)(7)(-2) = -56 < 0$

$$\frac{dp}{dt} = p' = p(3+p)(2-p)$$

$$\int \frac{dp}{p(3+p)(2-p)} = \int dt = t + C \qquad \text{partial fractions!}$$

$$\frac{1}{p(3+p)(2-p)} = \frac{A}{p} + \frac{B}{p+3} + \frac{C}{p-2} = \frac{A(p+3)(p-2)+Bp(p-2)+Cp(p+3)}{p(p+3)(p-2)}$$

$$\| $$

$$\frac{-1}{p(p+3)(p-2)} \qquad -1 = A(p+3)(p-2)+Bp(p-2)+Cp(p+3)$$

$p=-3$: $-1 = 0 + B(-3)(-5) + 0 = 15B$

$B = -1/15$

$p=2$: $-1 = 0 + 0 + C(2)(5) = 10C$

$C = -1/10$

$p=0$: $-1 = A(3)(-2) + 0 + 0 = -6A$

$A = 1/6$

$$\int \frac{1}{6}\frac{1}{p} - \frac{1}{15}\frac{1}{p+3} - \frac{1}{10}\frac{1}{p-2}\,dp = \frac{1}{6}\ln p - \frac{1}{15}\ln(p+3) - \frac{1}{10}\ln(p-2)$$

$$\boxed{\frac{1}{6}\ln p - \frac{1}{15}\ln(p+3) - \frac{1}{10}\ln(p-2) = t + C}$$

$$\ln\left(\frac{p^{1/6}}{(p+3)^{1/15}(p-2)^{1/10}}\right) = t + C \qquad \frac{p^{1/6}}{(p+3)^{1/15}(p-2)^{1/10}} = ke^t$$

denom

$t=0, p=3$: $\frac{3^{1/6}}{6^{1/15}\,1^{1/10}} = k \cdot 1 = k = \frac{3^{1/6}}{6^{1/15}}$

As $t \to \infty$, $ke^t \to \infty$ so $(p+3)^{1/15}(p-2)^{1/10} \to 0$

so $p \to -3$ or $p \to 2$. From the direction field, we conclude that $p \to 2$ as $t \to \infty$.