

Name:

Math 221 Section 3

Exam 2

Exams provide you, the student, with an opportunity to demonstrate your understanding of the techniques presented in the course. So:

Show all work. The steps you take to your answer are just as important, if not more important, than the answer itself. If you think it, write it!

1. (15 pts.) Find the solution to the initial value problem

$$2x^2y'' - xy' + y = 0 \quad y(1) = 2 \quad y'(1) = 5$$

$$y = x^r \quad y' = rx^{r-1}, \quad y'' = r(r-1)x^{r-2}$$

$$2r(r-1)x^r - rx^r + x^r = 0 = ((2r^2 - 2r) - r + 1)x^r \\ = (2r^2 - 3r + 1)x^r$$

$$2r^2 - 3r + 1 = 0 = (2r-1)(r-1) \rightarrow r = \frac{1}{2}, r = 1$$

$$y = c_1x^{\frac{1}{2}} + c_2x \quad y(1) = c_1 + c_2 = 2$$

$$y' = \frac{1}{2}c_1x^{-\frac{1}{2}} + c_2 \quad y'(1) = \frac{1}{2}c_1 + c_2 = 5 \quad c_1 + 2c_2 = 10$$

subtract: $c_2 = 8$; $c_1 = 2 - c_2 = 2 - 8 = -6$

$$y = -6x^{\frac{1}{2}} + 8x$$

2. (15 pts.) One solution to the differential equation

$$y'' - \frac{1}{t}y' + \frac{1-t}{t}y = 0$$

is $y_1(t) = e^t$.

Use reduction of order to find a second, independent, solution to the equation.

$$p(t) = -\frac{1}{t}$$

$$y_2 = c(t)y_1(t)$$

$$\int p(t) = -\ln t$$

$$c(t) = \int \frac{e^{-\int p(t) dt}}{(y_1(t))^2} dt = \int \frac{e^{\ln t}}{(e^t)^2} dt$$

$$= \int t e^{-2t} dt = \int t e^{-2t} dt \quad \begin{matrix} u=t & dv=e^{-2t} dt \\ du=dt & v=-\frac{1}{2}e^{-2t} \end{matrix}$$

$$= -\frac{1}{2}te^{-2t} - \frac{1}{4}e^{-2t}$$

$$\text{So } y_2(t) = \left(-\frac{1}{2}te^{-2t} - \frac{1}{4}e^{-2t}\right)e^t = \boxed{-\frac{1}{2}te^{-t} - \frac{1}{4}e^{-t}}$$

Check!

$$y_2' = -\frac{1}{2}e^{-t} + \frac{1}{2}te^{-t} + \frac{1}{4}e^{-t} = \frac{1}{2}te^{-t} - \frac{1}{4}e^{-t}$$

$$y_2'' = \frac{1}{2}e^{-t} - \frac{1}{2}te^{-t} + \frac{1}{4}e^{-t} = \frac{3}{4}e^{-t} - \frac{1}{2}te^{-t}$$

$$y_2'' - \frac{1}{t}y_2' + \left(\frac{1}{t}-1\right)y = \left(\frac{3}{4}e^{-t} - \frac{1}{2}te^{-t}\right) - \left(\frac{1}{2}e^{-t} - \frac{1}{4t}e^{-t}\right)$$

$$+ \left[\left(-\frac{1}{2}e^{-t} - \frac{1}{4t}e^{-t}\right) - \left(-\frac{1}{2}te^{-t} - \frac{1}{4}e^{-t}\right) \right]$$

$$= \left(\frac{3}{4} - \frac{1}{2} - \frac{1}{2} + \frac{1}{4}\right)e^{-t} + \left(-\frac{1}{2} + \frac{1}{2}\right)te^{-t} + \left(\frac{1}{4} - \frac{1}{4}\right)\frac{1}{t}e^{-t} = 0 + 0 + 0 = 0 \quad \checkmark!$$

3. (20 pts.) The functions

$$y_1(t) = t \text{ and } y_2(t) = te^t$$

form a fundamental set of solutions to the differential equation

$$y'' - \frac{t+2}{t}y' + \frac{t+2}{t^2}y = 0.$$

Use variation of parameters to find a particular solution to the equation

$$y'' - \frac{t+2}{t}y' + \frac{t+2}{t^2}y = t^3. = g(t)$$

$$y = c_1 y_1 + c_2 y_2 \quad \text{where} \quad \begin{aligned} c_1' y_1 + c_2' y_2 &= 0 \\ c_1' y_1' + c_2' y_2' &= g(t) \end{aligned}$$

$$\begin{aligned} c_1 &= \int \frac{-g y_2}{W} dt & c_2 &= \int \frac{g y_1}{W} dt & W &= \begin{vmatrix} t & te^t \\ 1 & te^t + e^t \end{vmatrix} \\ &= \int \frac{-t^3 te^t}{t^2 e^t} dt & &= \int \frac{t^3 \cdot t}{t^2 e^t} dt & &= te^t + te^t - te^t \\ &= \int -t^2 dt = -\frac{t^3}{3} & &= \int t^2 e^{-t} dt & &= t^2 e^t \\ & & & \text{--- } \begin{matrix} u=t^3 & dv=e^t dt \\ du=2t dt & v=-e^{-t} \end{matrix} & & \\ & & & = -t^2 e^{-t} + 2 \int te^{-t} dt & & \\ & & & \text{--- } \begin{matrix} u=t & dv=e^t dt \\ du=dt & v=-e^{-t} \end{matrix} & & \\ & & & = -t^2 e^{-t} + 2(-te^{-t} + \int e^{-t} dt) & & \\ & & & = -t^2 e^{-t} - 2te^{-t} - 2e^{-t} & & \end{aligned}$$

$$\begin{aligned} y &= \left(-\frac{t^3}{3}\right)(t) + (-t^2 e^{-t} - 2te^{-t} - 2e^{-t})(te^t) \\ &= -\frac{1}{4}t^4 - t^3 - 2t^2 - 2t \end{aligned}$$

4. (15 pts.) Find the general solution to the differential equation

$$y''' - 2y'' + 2y' - y = 2e^t - \sin t$$

$r=1?$
 $1-2+2-1=0 \checkmark$

$$y''' - 2y'' + 2y' - y = 0 \quad r^3 - 2r^2 + 2r - 1 = 0$$

$$(r-1)(r^2 - r + 1) = 0$$

$$r=1, r = \frac{1 \pm \sqrt{1-4}}{2} = \frac{1}{2} \pm \frac{\sqrt{3}}{2}i$$

$$y_1 = e^t, y_2 = e^{\frac{1}{2}t} \cos \frac{\sqrt{3}}{2}t, y_3 = e^{\frac{1}{2}t} \sin \frac{\sqrt{3}}{2}t$$

$$y'''' - 2y'' + 2y' - y = 2e^t$$

$$y'''' - 2y'' + 2y' - y = -\sin t$$

guess

~~$y = Ate^t$~~
 ~~$y = Ae^t$~~
 ~~$y = Ate^{2t}$~~
 ~~$y = Ae^{2t}$~~

$$y = Ate^t, y' = Ate^t + Ae^t$$

$$y'' = Ate^t + 2Ae^t \quad y''' = Ate^t + 3Ae^t$$

guess

$$y = A \sin t + B \cos t$$

$$y' = A \cos t - B \sin t$$

$$y'' = -A \sin t - B \cos t$$

$$y''' = -A \cos t + B \sin t$$

~~$Ate^t - 2Ate^t - 4Ae^t + 2Ate^t + 2Ae^t - Ate^t$~~

$$Ate^t + 3Ae^t - 2Ate^t - 4Ae^t + 2Ate^t + 2Ae^t - Ate^t$$

$$= (3A - 4A + 2A)e^t = Ae^t = 2e^t$$

$$\underline{A=2}$$

$$y'''' - 2y'' + 2y' - y = 1 \cdot \sin t + 0 \cdot \cos t$$

$$\begin{pmatrix} B \\ A+2A-2B-A \end{pmatrix} \sin t + \begin{pmatrix} -A \\ B+2B+2A-B \end{pmatrix} \cos t$$

$$= (A-B) \sin t + (A+B) \cos t$$

$$A-B = -1 \quad A+B = 0 \quad B = -A$$

$$2A = -1 \quad A = -\frac{1}{2}, B = \frac{1}{2}$$

$$y = c_1 e^t + c_2 e^{\frac{1}{2}t} \cos \frac{\sqrt{3}}{2}t + c_3 e^{\frac{1}{2}t} \sin \frac{\sqrt{3}}{2}t$$

$$+ 2e^t e^t + \frac{1}{2} \sin t + \frac{1}{2} \cos t$$

5. (15 pts.) Find the solution to the initial value problem

$$x' = 2x - 9y \quad x(0) = 1$$

$$y' = x + 2y \quad y(0) = -1$$

~~$$x(0) = 1 \quad x'(0) = 2(1) - 9(-1) = 11$$~~

$$y(0) = -1 \quad y'(0) = (1) + 2(-1) = -1$$

$$x = y' - 2y$$

$$y'' - 2y' = 2(y' - 2y) - 9y$$

$$y'' - 2y' = 2y' - 13y$$

$$y'' - 4y' + 13y = 0$$

$$r^2 - 4r + 13 = 0$$

$$r = \frac{4 \pm \sqrt{16 - 52}}{2} = \frac{4 \pm \sqrt{-36}}{2} = \frac{4 \pm 6i}{2}$$

$$= 2 \pm 3i$$

$$y(t) = c_1 e^{2t} \cos(3t) + c_2 e^{2t} \sin(3t)$$

$$y(0) = -1 = c_1 \cdot 1 \cdot 1 + c_2 \cdot 1 \cdot 0 = c_1 \quad c_1 = -1$$

$$y'(t) = 2c_1 e^{2t} \cos(3t) - 3c_1 e^{2t} \sin(3t) + 2c_2 e^{2t} \sin(3t) + 3c_2 e^{2t} \cos(3t)$$

$$-1 = y'(0) = 2c_1 - 0 + 0 + 3c_2$$

$$3c_2 = -1 - 2c_1 = -1 + 2 = 1$$

$$c_2 = \frac{1}{3}$$

$$y = -e^{2t} \cos(3t) + \frac{1}{3} e^{2t} \sin(3t)$$

$$x = y' - 2y = (-2e^{2t} \cos(3t) + 3e^{2t} \sin(3t) + \frac{2}{3} e^{2t} \sin(3t) + e^{2t} \cos(3t)) - 2(-e^{2t} \cos(3t) + \frac{1}{3} e^{2t} \sin(3t))$$

$$= (-2 + 1 + 2)e^{2t} \cos(3t) + (3 + \frac{2}{3} - \frac{2}{3})e^{2t} \sin(3t)$$

$$= e^{2t} \cos(3t) + 3e^{2t} \sin(3t)$$

6. (20 pts.) Sketch the direction field for the system of equations

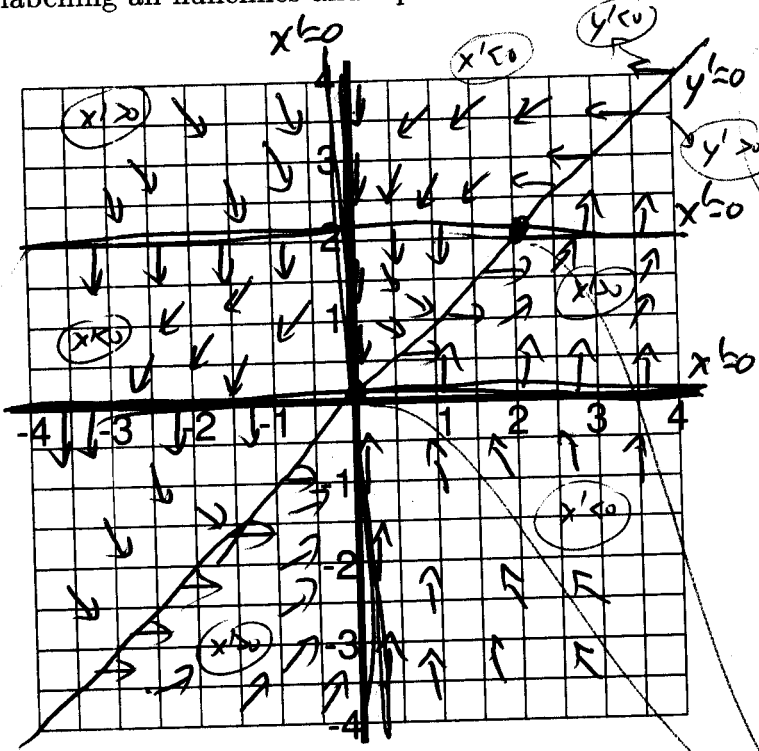
$$x' = 2xy - xy^2$$

$$y' = x - y$$

labelling all nullclines and equilibrium solutions.

$$x' = 2xy - xy^2 = 0$$

$$= xy(2-y)$$



$$\begin{matrix} x=0 \\ y=0 \\ y=2 \end{matrix} \quad \text{— vertical}$$

$$y' = x - y = 0$$

$$y=x \quad \text{horizontal}$$

equilibrium points