Seasonal variations in populations

Math 221 Spring 2004 Project

In this project, you will explore population models, based upon the logistic model

$$p' = ap - bp^2$$

where the coefficients a and b exhibit seasonal variations, i.e., are periodic, continuous, possibly non-constant functions of time t, with period = 1 year. (We choose a and b to be positive functions, to reflect the basic behaviors of typical populations.) Our goal is to understand the eventual long term behavior of the solutions, and how sensitive to initial conditions such solutions are.

Use Maple to plot the numerical solutions to the differential equation

$$(*) \quad \frac{dp}{dt} = a(t) \cdot p - b(t) \cdot p^2$$

for a variety of periodic functions of t with period 1:

(1) $a(t) = 3 + \sin(2\pi t)$, $b(t) = 4$
(2) $a(t) = 3$, $b(t) = 2 + \cos(2\pi t)$
(3) $a(t) = 2 + \cos(2\pi t)$, $b(t) = \dfrac{1 + \sin(4\pi t)}{2 + \sin(2\pi t)}$

plus two other pairs of periodic functions of your own choosing.

Plot solution curves for each, for several different initial conditions. What characteristics do your solutions have in common? How sensitive to the initial condition does the long-term behavior of the solution appear to be? What kinds of different behavior do you find for different choices of functions a, b ?

Boundedness of solutions: One of the distinguishing characteristics of the logistic model is that, unlike the Malthusian model, the population does not grow without bound. In the following you will show that our model, with periodic coefficients, maintains this basic characteristic:

Find the solution to (*) with $p(0) = 0$. Use this to show that if $p(t)$ solves (*) and $p(0) > 0$, then $p(t) > 0$ for all t.

Show that if $a(t) \leq a_0$ and $0 < b_0 \leq b(t)$ for all t, then $p'(t) < 0$ whenever $p(t) > \dfrac{a_0}{b_0}$. Explain why this implies that the graph of each solution to (*) will, for large enough t, lie below the line $p = \dfrac{a_0}{b_0} + 1$.

Finally, explain why these two facts together imply that solutions to our DE (*) always remain bounded for all time t.

Periodic solutions: When at least one of $a(t), b(t)$ are non-constant, then our equation (*) is typically neither linear nor separable, and so the methods we have developed in class will not help us to find an exact solution to the differential equation. In the case that $a(t) = a$ is constant, our equation

$$(**) \ \frac{dp}{dt} = a \cdot p - b(t) \cdot p^2$$

is, however, an example of a *Bernoulli equation.* We can solve this equation by rewriting it in terms of $1/p$, i.e., by setting $y = 1/p$, and then, using (**), determining what differential equation y satisfies. Show (by finding the relationship between $y'(t)$ and $p'(t)$) that y satisfies the equation

$$(***) \ \frac{dy}{dt} = -a \cdot y + b(t)$$

which we can (in principle) solve using the techniques we have learned.

Do this for example (2) above, finding the general solution for y (and hence p). Find an initial condition $p(0) = p_0$ where the solution $p(t)$ is *periodic* (with the same period as $b(t)$, i.e., $p(t+1) = p(t)$) . Compare this periodic solution to the numerical solutions you found above and the kinds of limiting behavior you observed. Find the maximum and minmum values that your periodic solution achieves, comparing them to the theoretical upper and lower bounds on the values that our analysis above found.

Turning back to the situation where $a(t) = a > 0$ is constant but $b(t)$ may be an arbitrary positive periodic function, we wish to demonstrate that the kind of behavior observed in this example is characteristic of the entire class (**) of equations. Show that a solution $y(t)$ to the equation (***), with $b(t)$ a periodic function with period 1, is periodic, provided that y(1)=y(0) (hint: show that $y(t)$ and $z(t) = y(t+1)$ satisfy the same initial value problem for (***)). Show, on the other hand, that there must always be an initial value $y(0) = y_0$ whose solution satisfies $y(1) = y_0$; we can use the fact that we know (in principle) how to explicitly write down solutions to

$$\frac{dy}{dt} + c(t) \cdot y = d(t)$$

as $y(t) = y_0 e^{-m(t)} + e^{-m(t)} \int_0^t e^{m(t)} d(t)\ dt$, where $m(t) = \int_0^t c(t)\ dt$, to solve the equation $y(1) = y_0$ for y_0 .

Explain why this allows us to conclude that our equation (**) always has a periodic solution.

Now that we have found a periodic solution to (***), call this solution y_p. This is a particular solution to the equation. Using the representation of the general solution to a linear DE, what can we conclude about the behavior of all other solutions to (***), in the long term? What does this tell us about the solutions to (**) in the long term?

Further considerations: When $a(t)$ is periodic but not constant, we cannot use the above techniques to carry out this kind of analysis. By experimenting with further plots, what kind of long-term behavior would we expect to see? Much of what we did also relied on the assuption that $a(t)$ and $b(t)$ were positive functions. What do experiments suggest would happen if we relaxed this assumption to allow for periodic functions which take negative values? Support your observations with graphs from your Maple output.

Plotting with Maple: The following commands work in Maple 9; for earlier implementations of Maple you may need to consult the documentation to determine the appropriate, slightly different, commands/syntax.

After starting up Maple, you first need to input:

```
with(DEtools):
```

To set up an IVP, we can define a few functions, e.g.,

```
f(t) := 1/(2+sin(t)); g(t) := 1*(2+sin(t));
```

define a differential equation

```
deqn := diff(y(t),t)=f(t)*y(t)-g(t)*y(t)*y(t);
```

and set an initial condition or two or ten,

```
init1 := y(0)=1; init2 := y(0)=2;
```

Then to plot a family of solutions, we can type

```
DEplot(deqn,y(t),t=0..10,[[init1],[init2]],y=0..2,stepsize=.05,linecolor=black);
```

These commands can be adjusted to suit the particular IVP's you wish to solve; to focus in on various parts of the graphs of the solutions, for example, we can adjust the range of t's and y's (e.g., `t=1.5..2.1` and `y=1.3..1.5`) which Maple plots. Note: Maple uses the name `Pi` to denote the number π .

Final project report: Your final project report should be an organized discussion of the problem, with an introduction and a conclusion. It should include the plots you create in Maple, with clear labels. You should explain your results and conclusions, using your own words. Assume that your reader is someone who took a differential equations course a while ago and does not remember all of the details. In particular, you should explain any ideas you need from our class discussions on first-order ODEs. The grade for this project will be based on both your results *and* on your explanations.

Deadline: Your final report is due on Tuesday, April 13, at the start of class.