Name: Solution

Math 221, Section 3

Quiz number 1

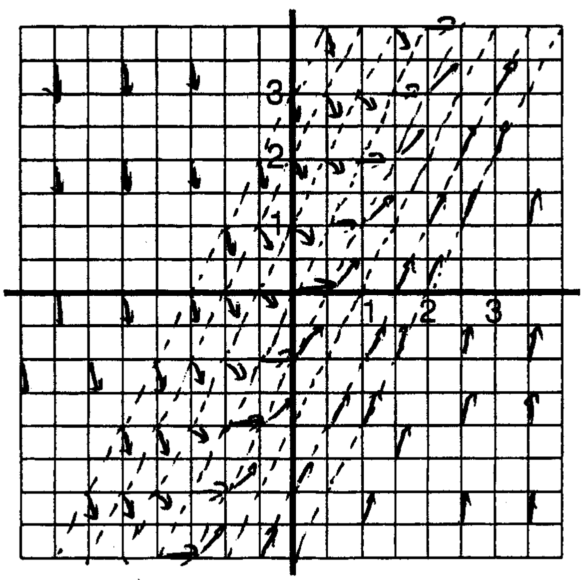
Show all work. How you get your answer is just as important, if not more important, than the answer itself. If you think it, write it!

1. Use isoclines to sketch the direction field for the differential equation

$$\frac{dy}{dx} = 2x - y$$

(draw in the isoclines as dotted curves), and sketch the solutions which pass through the points

$$(0,0)\ ,\ (0,2)\ ,\ \text{and}\ (0,-2)\ .$$



$y' = 2x - y = c =$ constant

$\boxed{y = 2x - c}$

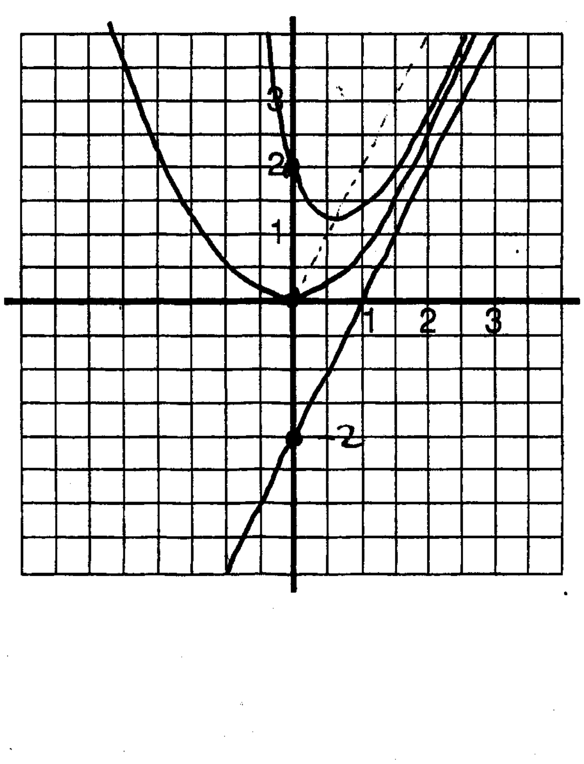
$c=0 \quad y = 2x$

$c=1 \quad y = 2x-1$

$c=2 \quad y = 2x-2$

$c=-1 \quad y = 2x+1$

$c=-2 \quad y = 2x+2$



Name: Solution

Math 221, Section 3

Quiz number 2

Show all work. How you get your answer is just as important, if not more important, than the answer itself. If you think it, write it!

Find the (implicit) solutions to the differential equation

$$\frac{dy}{dx} = \frac{x^3+x}{y\cos y}$$

and find the specific solution satisfying the additional initial value condition

$$y(2) = 0\ .$$

$$y\cos y \frac{dy}{dx} = x^3 + x$$

$$\int y\cos y\, dy = \int x^3 + x\, dx = \frac{x^4}{4} + \frac{x^2}{2} + C$$

$u = y \quad dv = \cos y\, dy$

$du = dy \quad v = \sin y$

$$= y\sin y - \int \sin y\, dy = y\sin y + \cos y$$

$$\boxed{y\sin y + \cos y = \frac{x^4}{4} + \frac{x^2}{2} + C}$$

$x=2 \quad y=0$: $\quad 0\cdot\sin(0) + \cos(0) = \frac{2^4}{4} + \frac{2^2}{2} + C$

$$0 + 1 = 4 + 2 + C$$

$$C = 1 - 6 = -5$$

$$\boxed{y\sin y + \cos y = \frac{x^4}{4} + \frac{x^2}{2} - 5}$$

Solution

Name:

Math 221, Section 3

Quiz number 3

Show all work. How you get your answer is just as important, if not more important, than the answer itself. If you think it, write it!

Find the solution to the initial value problem

$$\frac{dy}{dx} = \frac{3}{x}y + x^2 \ln x$$
$$y(2) = 2\,.$$

$p(x) = -3/x$

$g(x) = x^2 \ln x$

$y' = \frac{3}{x}y + x^2 \ln x \qquad y' + \left(\frac{-3}{x}\right)y = x^2 \ln x$

$\int p(x)\,dx = \int \frac{-3}{x}dx = -3\int \frac{dx}{x} = -3\ln x \qquad e^{\int p(x)dx} = e^{-3\ln x} = e^{\ln(x^{-3})} = x^{-3}$

$u = \ln x$

$du = \frac{1}{x}dx$

$\int g(x)e^{\int p(x)dx}dx = \int x^2 \ln x (x^{-3})\,dx = \int \frac{\ln x}{x}dx$

$= \int u\,du\Big|_{u=\ln x} = \frac{u^2}{2}\Big|_{u=\ln x} = \frac{(\ln x)^2}{2}$

$y = e^{-\int p(x)dx}\left(\frac{(\ln x)^2}{2} + C\right) = x^3\left(\frac{(\ln x)^2}{2} + C\right) = \frac{1}{2}x^3(\ln x)^2 + Cx^3$

$2 = y(2) = \frac{1}{2}(8)(\ln 2)^2 + C(8) = 4(\ln 2)^2 + 8C$

$8C = 2 - 4(\ln 2)^2 \qquad C = \frac{1}{8}(2 - 4(\ln 2)^2) = \frac{1}{4} - \frac{1}{2}(\ln 2)^2$

$$\boxed{y = \frac{1}{2}x^3(\ln x)^2 + \left(\frac{1}{4} - \frac{1}{2}(\ln 2)^2\right)x^3}$$

Solution

Name:

Math 221, Section 3

Quiz number 4

Show all work. How you get your answer is just as important, if not more important, than the answer itself. If you think it, write it!

After a long night of studying, you discover that you at some point left your Math 221 textbook in your 35° refrigerator. Curious to figure out when it was that you did this, you note that the room temperature (and so the book's original temperature) is 75°, that the book's current temperature is 40°, and, 15 minutes later, it's temperature is 38°. How long before your first temperature measurement did your book go into the fridge?

$A = 35$
$T(0) = 40$
$T(15) = 38$

When is $T(t) = 75$?

$T' = k(A - T)$ $\quad T' + kT = kA$

$T(t) = e^{-\int k\,dt}\left(\int kA e^{\int k\,dt} + C\right) = e^{-kt}\left(Ae^{kt} + C\right)$

$= A + Ce^{-kt} = 35 + Ce^{-kt}$

$T(0) = 40 = 35 + C$, $C = 5$, $T(t) = 35 + 5e^{-kt}$

$38 = T(15) = 35 + 5e^{-15k}$, $3 = 5e^{-15k}$, $\frac{3}{5} = e^{-15k}$

$\ln(\frac{3}{5}) = -15k$ $\quad k = \frac{-1}{15}\ln(\frac{3}{5})$

$\boxed{T(t) = 35 + 5e^{\frac{t}{15}\ln(\frac{3}{5})}}$

When is $T(t) = 75$?

$75 = 35 + 5e^{\frac{t}{15}\ln(\frac{3}{5})}$

$40 = 5e^{\frac{t}{15}\ln(\frac{3}{5})}$, $8 = e^{\frac{t}{15}\ln(\frac{3}{5})}$ $\quad \ln(8) = \frac{t}{15}\ln(\frac{3}{5})$

$t = \dfrac{15\ln(8)}{\ln(3/5)}$ (<0) is when the book went into the fridge.

Solutions

Name:

Math 221, Section 3

Quiz number 5

Show all work. How you get your answer is just as important, if not more important, than the answer itself. If you think it, write it!

Show that $y_1(x) = \sec x$ and $y_2(x) = \tan x$ are a fundamental set of solutions to the second order equation

$$y'' - (\tan x)y' - (\sec^2 x)y = 0$$

Use this to find the solution to the initial value problem

$$y'' - (\tan x)y' - (\sec^2 x)y = 0 \qquad y(\pi/4) = 1, y'(\pi/4) = 4$$

$y_1 = \sec x$ $\quad y_2 = \tan x$ $\quad y_1(\frac{\pi}{4}) = \sqrt{2}$ $\quad y_2(\frac{\pi}{4}) = 1$

$y_1' = \sec x \tan x$ $\quad y_2' = \sec^2 x$ $\quad y_1'(\frac{\pi}{4}) = \sqrt{2}$ $\quad y_2'(\frac{\pi}{4}) = 2$

$y_1'' = \sec x \tan^2 x + \sec^3 x$ $\quad y_2'' = 2\sec^2 x \tan x$

Check they are solutions:

$(\sec x \tan^2 x + \sec^3 x) - (\tan x)(\sec x \tan x) - (\sec^2 x)(\sec x) = 0$ ✓

$(2\sec^2 x \tan x) - (\tan x)(\sec^2 x) - (\sec^2 x)(\tan x) = (2-1-1)\sec^2 x \tan x = 0$ ✓

Fundamental set of solutions?

$$W(y_1, y_2) = \begin{vmatrix} \sec x & \tan x \\ \sec x \tan x & \sec^2 x \end{vmatrix} = \sec^3 x - \sec x \tan^2 x = \sec x(\sec^2 x - \tan^2 x) = \sec x$$

So e.g. $W(y_1,y_2)(\frac{\pi}{4}) = \sec(\frac{\pi}{4}) = \sqrt{2} \neq 0$, so yes. ✓

Solution to IVP: $y = c_1 \sec x + c_2 \tan x$ where

$$c_1 = \frac{\begin{vmatrix} 1 & 1 \\ 4 & 2 \end{vmatrix}}{\begin{vmatrix} \sqrt{2} & 1 \\ \sqrt{2} & 2 \end{vmatrix}} = \frac{2-4}{2\sqrt{2}-\sqrt{2}} = \frac{-2}{\sqrt{2}} = -\sqrt{2}$$

$$c_2 = \frac{\begin{vmatrix} \sqrt{2} & 1 \\ \sqrt{2} & 4 \end{vmatrix}}{\begin{vmatrix} \sqrt{2} & 1 \\ \sqrt{2} & 2 \end{vmatrix}} = \frac{4\sqrt{2}-\sqrt{2}}{2\sqrt{2}-\sqrt{2}} = \frac{3\sqrt{2}}{\sqrt{2}} = 3$$

So $\boxed{y = -\sqrt{2}\sec x + 3\tan x}$

Solution

Name:

Math 221, Section 3

Quiz number 6

Show all work. How you get your answer is just as important, if not more important, than the answer itself. If you think it, write it!

1. Find the solution to the initial value problem

$$2y'' - 7y' + 3y = 0$$

$$y(0) = 1 \quad , \quad y'(0) = -7$$

Auxiliary eqn: $2r^2 - 7r + 3 = 0$

$$r = \frac{7 \pm \sqrt{49 - 4 \cdot 6}}{4} = \frac{7 \pm \sqrt{25}}{4} = \frac{7 \pm 5}{4}$$

$$= \frac{1}{2}, 3$$

Fundamental set of solutions: $y_1 = e^{\frac{1}{2}x}$, $y_2 = e^{3x}$

$$y = c_1 e^{\frac{1}{2}x} + c_2 e^{3x}$$

$$y' = \tfrac{1}{2} c_1 e^{\frac{1}{2}x} + 3c_2 e^{3x}$$

$y(0) = 1 = c_1 + c_2$

$y'(0) = -7 = \frac{1}{2} c_1 + 3c_2$

$c_1 + c_2 = 1$

$\frac{1}{2} c_1 + 3c_2 = -7 \qquad c_1 + 6c_2 = -14$

$5c_2 = -15 \qquad c_2 = -3$

$c_1 = 1 - c_2 = 1 - (-3) = 4$

$$\boxed{y = 4e^{\frac{1}{2}x} - 3e^{3x}}$$

Solution

Name:

Math 221, Section 3

Quiz number 7

Show all work. How you get your answer is just as important, if not more important, than the answer itself. If you think it, write it!

1. Find the general solution to the differential equation
$$y'' - y' + y = x^2 - 1 + e^x \sin x$$

$y''-y'+y=0 \qquad r^2-r+1=0 \qquad r=\frac{1\pm\sqrt{1-4}}{2}=\frac{1}{2}\pm\frac{\sqrt{3}}{2}i$

$y_1=e^{\frac{1}{2}x}\cos\frac{\sqrt{3}}{2}x \;,\; y_2=e^{\frac{1}{2}x}\sin\frac{\sqrt{3}}{2}x$

$y''-y'+y=x^2-1$

$y=ax^2+bx+c$

$y'=2ax+b$

$y''=2a$

$y''-y'+y=2a-2ax-b+ax^2+bx+c$

$=ax^2+(b-2a)x+(c-b+2a)$

$=1\cdot x^2+0\cdot x+(-1)$

$a=1$

$b-2a=b-2=0,\; b=2$

$c-b+2a=c-2+2=-1,\; c=-1$

$y=x^2+2x-1$

$y''-y'+y=e^x\sin x$

$y=Ae^x\sin x+Be^x\cos x$

$y'=Ae^x\sin x+Ae^x\cos x+Be^x\cos x$
$\quad -Be^x\sin x$

$=(A-B)e^x\sin x+(A+B)e^x\cos x$

$y''=(A-B)e^x\sin x+(A-B)e^x\cos x$
$\quad +(A+B)e^x\cos x-(A+B)e^x\sin x$

$=-2Be^x\sin x+2Ae^x\cos x$

$y''-y'+y=(-2B-A+B+A)e^x\sin x$
$\quad +(2A-A-B+B)e^x\cos x$

$=-Be^x\sin x+Ae^x\cos x$

$=e^x\sin x \rightsquigarrow B=-1,\; A=0$

$y=-e^x\cos x$

Gen'l solution:

$$\boxed{y=c_1e^{\frac{1}{2}x}\cos\frac{\sqrt{3}}{2}x+c_2e^{\frac{1}{2}x}\sin\frac{\sqrt{3}}{2}x+(x^2+2x-1)+(-e^x\cos x)}$$

Solution

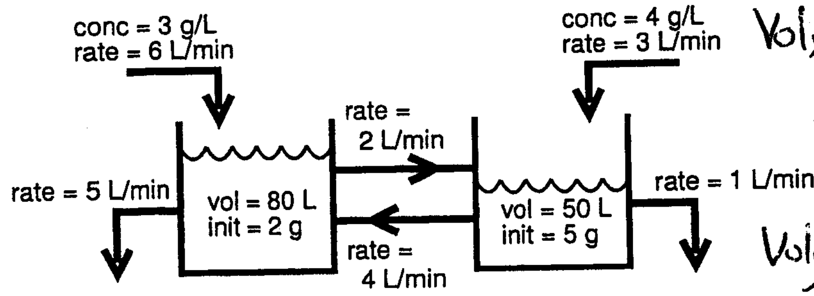
Name:

Math 221, Section 3

Quiz number 8

Show all work. How you get your answer is just as important, if not more important, than the answer itself. If you think it, write it!

1. The figure below represents a pair of tanks, with salt in solution flowing in and out and between them with the specified rates; the pipes flowing in are marked with their concentrations, and the tanks are marked with their initial amounts of salt and initial volumes. Set up the system of equations which describe the rates at which the two amounts of salt will change over time, and show how to convert this system to a single higher order initial value problem. *You do* not *need to solve this IVP.*



$$\mathrm{Vol}_x = 80 + (6+4-2-5)t = 80+3t$$

$$\mathrm{Vol}_y = 50 + (3+2-4-1)t = 50$$

$$x' = \text{in} - \text{out} = 3\cdot 6 + 4\cdot\frac{y}{50} - 5\cdot\frac{x}{80+3t} - 2\frac{x}{80+3t}$$

$$y' = \text{in} - \text{out} = 4\cdot 3 + 2\frac{x}{80+3t} - 4\frac{y}{50} - 1\cdot\frac{y}{50}$$

$$\boxed{\begin{aligned} x' &= 18 + \frac{4}{50}y - \frac{7}{80+3t}x \\ y' &= 12 + \frac{2}{80+3t}x - \frac{5}{50}y \end{aligned}} \longleftrightarrow \text{IVP} \longleftrightarrow \boxed{\begin{aligned} x(0) &= 2 \\ y(0) &= 5 \end{aligned}}$$

$$y = \frac{50}{4}\left(x' + \frac{7}{80+3t}x - 18\right)$$

$$\boxed{\frac{50}{4}\left(x'' + \frac{7}{80+3t}x' + \left(\frac{-7}{(80+3t)^2}\cdot 3\right)x\right) = 12 + \frac{2}{80+3t}x - \frac{5}{50}\left(\frac{50}{4}\left(x' + \frac{7}{80+3t}x - 18\right)\right)}$$

$$\boxed{\begin{aligned} x(0) &= 2 \\ x'(0) &= 18 - \frac{4}{50}\cdot 5 + \frac{7}{80}\cdot 2 \end{aligned}}$$

2nd order IVP in $\underline{x}$.

Solution

Name:

Math 221, Section 3

Quiz number 9

Show all work. How you get your answer is just as important, if not more important, than the answer itself. If you think it, write it!

Find the general solution to the system of equations

$$x' = 2x + y$$
$$y' = 6x + y$$

and determine which of the nine basic behaviors best characterizes the phase portrait of the set of solutions.

$y = x' - 2x \qquad x'' - 2x' = 6x + x' - 2x = x' + 4x$

$x'' - 3x' - 4x = 0 \qquad r^2 - 3r - 4 = 0 = (r-4)(r+1)$

$r = 4, -1$

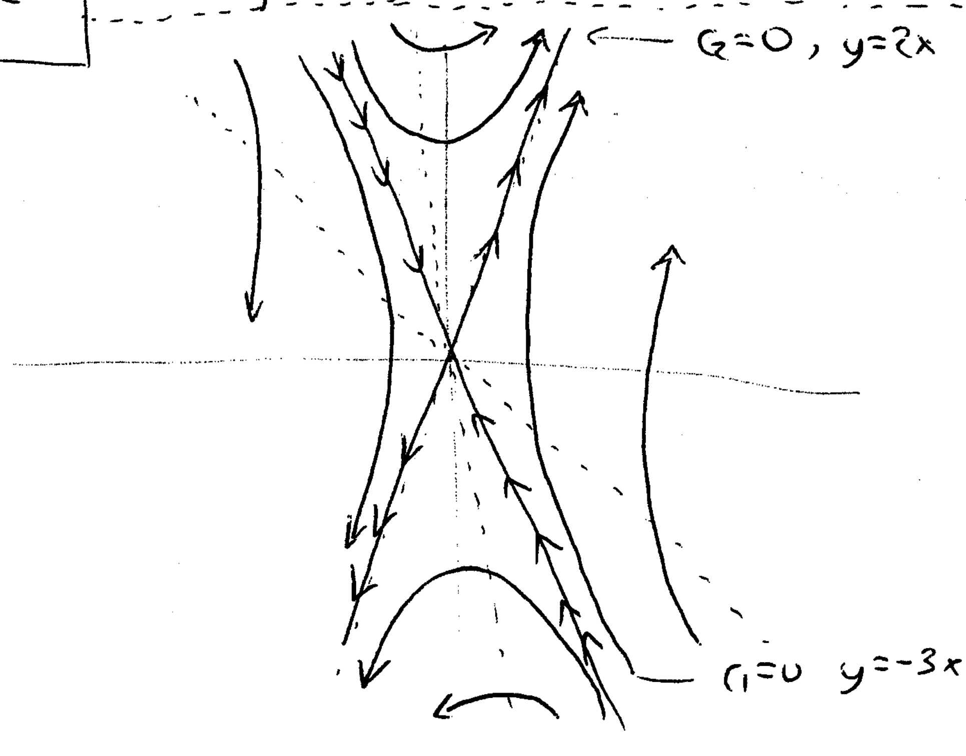
$x = c_1 e^{4t} + c_2 e^{-t}$

$y = x' - 2x = 4c_1 e^{4t} - c_2 e^{-t} - 2c_1 e^{4t} - 2c_2 e^{-t} = 2c_1 e^{4t} - 3c_2 e^{-t}$

$$\boxed{\begin{aligned} x &= c_1 e^{4t} + c_2 e^{-t} \\ y &= 2c_1 e^{4t} - 3c_2 e^{-t} \end{aligned}}$$

Distinct real roots, opposite sign

$\Rightarrow$ phase portrait is a saddle around $(0,0)$



Name: Solutions

Math 221, Section 3

Quiz number 10

Show all work. How you get your answer is just as important, if not more important, than the answer itself. If you think it, write it!

Use Laplace tansforms to find the solution to the initial value problem

$$y' - 3y = 5t$$
$$y(0) = 2$$

$$\mathcal{L}\{y'-3y\} = \mathcal{L}\{5t\} = 5\mathcal{L}\{t\} = 5\frac{1}{s^2}$$

$$\parallel$$

$$(s\mathcal{L}\{y\}-y(0)) - 3\mathcal{L}\{y\} = (s-3)\mathcal{L}\{y\} - 2$$

$$(s-3)\mathcal{L}\{y\} = 2 + 5\frac{1}{s^2} \qquad \mathcal{L}\{y\} = 2\frac{1}{s-3} + 5\frac{1}{s^2(s-3)}$$

$$\frac{1}{s^2(s-3)} = \frac{A}{s} + \frac{B}{s^2} + \frac{C}{s-3} = \frac{As(s-3) + B(s-3) + Cs^2}{s^2(s-3)}$$

$$1 = As(s-3) + B(s-3) + Cs^2$$

$s=0$: $1 = 0 + B(-3) + 0 = -3B$; $B = -1/3$

$s=3$: $1 = 0 + 0 + C(9) = 9C$; $C = 1/9$

$s=1$: $1 = A(-2) + B(-2) + C = -2A + 2/3 + 1/9$

$-2A = 1 - 2/3 - 1/9 = 2/9$; $A = -1/9$

$$\mathcal{L}\{y\} = 2\frac{1}{s-3} + \frac{-5}{9}\frac{1}{s} - \frac{5}{3}\frac{1}{s^2} + \frac{5}{9}\frac{1}{s-3} = \frac{-5}{9}\frac{1}{s} - \frac{5}{3}\frac{1}{s^2} + \frac{23}{9}\frac{1}{s-3}$$

$$y = -\frac{5}{9}\mathcal{L}^{-1}\left\{\frac{1}{s}\right\} - \frac{5}{3}\mathcal{L}^{-1}\left\{\frac{1}{s^2}\right\} + \frac{23}{9}\mathcal{L}^{-1}\left\{\frac{1}{s-3}\right\}$$

$$= \frac{-5}{9}\cdot 1 - \frac{5}{3}\cdot t + \frac{23}{9}e^{3t}$$

$$\boxed{y = \frac{23}{9}e^{3t} - \frac{5}{9} - \frac{5}{3}t}$$