Math 221

Topics since the second exam

Laplace Transforms.

There is a whole different set of techniques for solving n-th order linear equations, which are based on the *Laplace transform* of a function. For a function $f(t)$, it's Laplace transform is

$$\mathcal{L}\{f\} = \mathcal{L}\{f\}(s) = \int_0^\infty e^{-st} f(t)\ dt$$

The domain of $\mathcal{L}\{f\}$ is all values of s where the improper integral converges. For most basic functions f, $\mathcal{L}\{f\}$ can be computed by integrating by parts. A list of such transforms can be found on the handout from class. The most important property of the Laplace transform is that it *turns differentiation into multiplication by s.* that is:

$$\mathcal{L}\{f'\}(s) = s\mathcal{L}\{f\}(s) - f(0)$$

more generally, for the n-th derivative:

$$\mathcal{L}\{f^{(n)}\}(s) = s^n\mathcal{L}\{f\}(s) - s^{n-1}f(0) - s^{n-2}f'(0) - \cdots - f^{(n-1)}(0)$$

The Laplace transform is a *linear operator* in the same sense that we have used the term before: for any functions f and g, and any constants a and b,

$$\mathcal{L}\{af + bg\} = a\mathcal{L}\{f\} + b\mathcal{L}\{g\}$$

(since integration is a linear operator). We can therefore use Laplace transforms to solve linear (inhomogeneous) equations (with constant coefficients), by applying $\mathcal{L}$ to both sides of the equation:

$$ay'' + by' + cy = g(t)$$

becomes

$$(as^2 + bs + c)\mathcal{L}\{y\} - asy(0) - ay'(0) - by(0) = \mathcal{L}\{g\}, \text{ i.e.}$$

$$\mathcal{L}\{y\} = \frac{\mathcal{L}\{g\}(s) + asy(0) + ay'(0) + by(0)}{as^2 + bs + c}$$

So to solve our original equation, we need to find a function y whose Laplace transform is this function on the right. It turns out there is a formula (involving an integral) for the *inverse Laplace transform* $\mathcal{L}^{-1}$, which in principle will solve our problem, but the formula is too complicated to use in practice. Instead, we will develop techniques for recognizing functions as linear combinations of the functions appearing as the right-hand sides of the formulas in our Laplace transform tables. Then the function y we want is the corresponding combination of the functions on the left-hand sides of the formulas, because the Laplace transform is linear! Note that this approach incorporates the initial value data $y(0), y'(0)$ into the solution; it is naturally suited to solving initial value problems.

Our basic technique for finding solutions is *partial fractions*: we will content ourselves with a simplified form of it, sufficient for solving second order equations. The basic idea is that we need to find the inverse Laplace transform of a function having a quadratic polynomial as^2+bs+c in its denomenator. Partial fractions tells us that, if we can factor as^2+bs+c $= a(x-r_1)(x-r_2)$, where $r_1 \neq r_2$, then any function

$$\frac{ms+n}{as^2+bs+c}=\frac{A}{s-r_1}+\frac{B}{s-r_2}$$

for appropriate constants A and B. We can find the constants by writing

$$\frac{A}{s-r_1}+\frac{B}{s-r_2}=\frac{A(s-r_2)+B(s-r_1)}{(s-r_1)(s-r_2)}=\frac{Aa(s-r_2)+Ba(s-r_1)}{as^2+bs+c}$$

so we must have $ms+n=Aa(s-r_2)+Ba(s-r_1)$; setting the coefficients of the two linear functions equal to one another, we can solve for A and B. We can therefore find the inverse Laplace transform of $(ms+n)/(as^2+bs+c)$ as a combination of the inverse transforms of $(s-r_1)^{-1}$ and $(s-r_2)^{-1}$, which can be found on the tables!

If $r_1=r_2$, then we instead write

$$\frac{ms+n}{as^2+bs+c}=\frac{A}{s-r_1}+\frac{B}{(s-r_1)^2}=\frac{a(A(s-r_1)+B)}{a(s-r_1)^2}=\frac{a(A(s-r_1)+B)}{as^2+bs+c}$$

anbd solve for A and B as before.

Finally, if we *cannot* factor as^2+bs+c (i.e, it has complex roots), we can then write it as (a times) a sum of squares, by completing the square:

$as^2+bs+c=a((s-\alpha)^2+\beta^2)$, so

$$\frac{ms+n}{as^2+bs+c}=\frac{A\beta}{a((s-\alpha)^2+\beta^2)}+\frac{B(s-\alpha)}{a((s-\alpha)^2+\beta^2)}=$$

$$\frac{A}{a}\frac{\beta}{(s-\alpha)^2+\beta^2}+\frac{B}{a}\frac{s-\alpha}{(s-\alpha)^2+\beta^2}$$

for appropriate constants A and B (which we solve for by equating the numerators), and so it is a linear combination of $\dfrac{\beta}{(s-\alpha)^2+\beta^2}$ and $\dfrac{(s-\alpha)}{(s-\alpha)^2+\beta^2}$, both of which appear on our tables!

Handling higher degree polynomials in the denomenator is similar; if all roots are real and distinct, we write our quotient as a linear combination of the functions $(s-r_i)^{-1}$, combine into a single fraction, and set the numerators equal; if we have repeated roots, we include terms in the sum with successively higher powers $(s-r_i)^{-k}$ (where k runs from 1 to the multiplicity of the root). Complex roots are handled by inserting the term we dealt with above into the sum.

Discontinuous external force.

One area in which Laplace transforms provide a better framework for working out solutions than our "auxiliary equation" approach is when we are trying to solve an equation

$$ay''+by'+cy=g(t)$$

where $g(t)$ is *discontinuous*, or defined in *pieces* over different time intervals. The model for a discontinuous function is the step function $u(t)$:

$$u(t)=\begin{cases}1 & \text{if } t\geq 0\\ 0 & \text{if } t<0\end{cases}$$

More generally, the function $u(t-a)$ has

$$u(t-a)=\begin{cases}1 & \text{if } t\geq a\\ 0 & \text{if } t<a\end{cases}$$

So, for example, the function which is t for $3\leq t\leq 5$, and is 0 everywhere else, can be expressed as $g(t)=t(u(t-3)-u(t-5))$. We can streamline things somewhat by writing $u(t-a)-u(t-b)=\chi_{[a,b]}(t)=$ the characteristic function of the interval $[a,b]$; it is 1 between a and b, and 0 everywhere else. So, for example, the piecewise-defined function

$$f(t)=\begin{cases}t & \text{if } 0\leq t\leq 2\\ 5-t & \text{if } 2<t<5\\ 3 & \text{if } t\geq 5\end{cases}$$

can be expressed as

$$\begin{aligned}f(t)&=t\chi_{[0,2]}(t)+(5-t)\chi_{[2,5]}(t)+3\chi_{[5,\infty)}(t)\\ &=t(u(t)-u(t-2))+(5-t)(u(t-2)-u(t-5))+3u(t-5)\end{aligned}$$

We can find the Laplace transform of such a function by finding the transform of functions of the form $f(t)u(t-a)$, which we can do directly from the integral, by making the substitution $x=t-a$:
$\mathcal{L}\{f(t)u(t-a)\}=\int_0^\infty e^{-st}f(t)u(t-a)\ dt=\int_a^\infty e^{-st}f(t)\ dt=\int_0^\infty e^{-s(t+a)}f(t+a)\ dt=e^{-as}\int_0^\infty e^{-st}f(t+a)\ dt=e^{-as}\mathcal{L}\{f(t+a)\}$.
Turning this around, we find that the inverse Laplace transform of the function $e^{-as}\mathcal{L}\{f\}(s)$ is $f(t-a)u(t-a)$. So if we can find the inverse transform of a function $F(s)$ (in our tables), this tells us how to find the inverse transform of $e^{-as}F(s)$. This is turn gives us a method for solving any initial value problem, in principle, whose inhomogeneous term $f(t)$ has finitely many values where it is discontinuous, by writing $f(t)$ as a sum of functions of the form $f_i(t)u(t-a_i)$, as above.

For example, to find the solution to the differential equation

$y''+2y'+5y=g(t)$, $y(0)=2$, $y'(0)=1$, where $g(t)$ is the function which is 5 for $2\leq t\leq 4$ and 0 otherwise, we would (after taking Laplace transforms and simplifying) need to find the inverse Laplace transform of the function

$$F(s)=\frac{2s+5}{s^2+2s+5}+\frac{5(e^{-2s}-e^{-4s})}{s(s^2+2s+5)}$$

Applying our partial fractions techniques, we find that

$$\begin{aligned}F(s)=2\frac{s+1}{(s+1)^2+2^2}+\frac{3}{2}\frac{2}{(s+1)^2+2^2}+\left(\frac{1}{s}-\frac{s+1}{(s+1)^2+2^2}\frac{1}{2}-\frac{2}{(s+1)^2+2^2}\right)e^{-2s}\\ -\left(\frac{1}{s}-\frac{s+1}{(s+1)^2+2^2}-\frac{1}{2}\frac{2}{(s+1)^2+2^2}\right)e^{-4s}\end{aligned}$$

We can apply $\mathcal{L}^{-1}$ to each term, using $\mathcal{L}^{-1}\{e^{-as}\mathcal{L}\{f\}(s)\}=f(t-a)u(t-a)$ for the last 6 terms (since after removing e^{-2s} and e^{-4s} the remainder of each term is in our tables). For example,

$\mathcal{L}^{-1}\{\frac{s+1}{(s+1)^2+2^2}e^{-2s}\}=e^{-(t-2)}\cos(2(t-2))u(t-2)$. The final solution, as the interested reader can work out, is

$$
\begin{aligned}
y = 2e^{-t}\cos(2t) + \frac{3}{2}e^{-t}\sin(2t) \\
+ \left[1 - e^{-(t-2)}\cos(2(t-2)) - \frac{1}{2}e^{-(t-2)}\sin(2(t-2))\right] u(t-2) \\
- \left[1 - e^{-(t-4)}\cos(2(t-4)) - \frac{1}{2}e^{-(t-4)}\sin(2(t-4))\right] u(t-4)
\end{aligned}
$$

The Dirac delta function and abrupt changes in direction

One final type of forcing term to which the method of Laplace transforms is especially well adapted is an *impulsive force*, where the velocity of the object instantaneously changes. We can model this as an external force $F(t)$ which applies a very large force over a very short time, so the the *impulse* it imparts over the time interval $(-\epsilon, \epsilon)$, given by

$\int_{-\epsilon}^{\epsilon} F(t)\ dt = 1$

This requires $F(t)$ to be very large! If we imagine taking a limit as the length of the interval goes to zero, we get the Dirac delta function:

$\delta(t) = \begin{cases} 0 & \text{if } t \neq 0 \\ \infty & \text{if } t = 0 \end{cases}$ with $\int_{-\epsilon}^{\epsilon} \delta(t)\ dt = 1$ for every ϵ .

One consequence of the integral equality is that, for any function $f(t)$,

$\int_{-\infty}^{\infty} f(t)\delta(t)\ dt = f(0)$

The only problem is that $\delta(t)$ isn't a function! (∞ can't be the "value" at a point...) What it is is a "generalized function" (or *distribution*). We won't go into further details, but with the proper framework we can work with it as we do ordinary functions. Which is good, because a force of $\delta(t)$ is precisely the kind of function we need to model a hammer blow; it essentially instantly changes the motion of an object. From the pointof view of an impulse imparted to the object, integrating a force integrated my'', which gives $my'(b) - my'(a)$, which is a change in momentum. So a Dirac delta, which represents an instantaneous impulse, represents an instant change of momentum, and so an instant jump in velocity.

But computing the Laplace transform of $\delta(t)$, or rather $\delta(t-a)$ (which represents a hammer blow at time a) is straighforward.

$\mathcal{L}\{\delta(t-a)\} = \int_0^{\infty} e^{-st}\delta(t-a)\ dt = \int_0^{\infty} e^{-s(u+a)}\delta(u)\ du = \int_{-\infty}^{\infty} e^{-s(u+a)}\delta(u)\ du = e^{-as}$

where the second equality uses a u-substitution. With this in hand, we can solve IVP's having a forcing term that includes an impulsive (= delta function) forcing term, using the same approach we have with our other IVP's. For example, to solve

$y'' + y = 3\chi_{[2,4]} - 2\delta(t-5) \quad y(0) = 2\ ,\ y'(0) = 1$

(i.e., a rocket firing for $2 \leq t \leq 4$, followed by an (upward) hammer blow at $t = 5$), we solve

$(s^2+1)\mathcal{L}\{y\} = 3\left(\frac{e^{-2s}}{s} - \frac{e^{-4s}}{s}\right) - 2e^{-5s}$

so $\mathcal{L}\{y\} = 3\left(\frac{e^{-2s}}{s(s^2+1)} - \frac{e^{-4s}}{s(s^2+1)}\right) - 2\frac{e^{-5s}}{s^2+1}$

so $y = 3u(t-2)f(t-2) - 3u(t-4)f(t-4) - 2u(t-5)g(t-5)$, where
$f(t) = \mathcal{L}^{-1}\left\{\frac{1}{s(s^2+1)}\right\}$ and $g(t) = \mathcal{L}^{-1}\left\{\frac{1}{s^2+1}\right\} = \sin t$. We can determine f by partial fractions.

And, just because it managed to be left off of the Exam 2 topics sheet:

Applications: spring - mass problems

Basic setup: an object with mass m sits on a track and is attached to an immovable wall by a spring. At rest, the mass sits at a point in the track which we will call 0. The mass is then displaced from this equilibrium position and released (with some initial velocity). The position at time t of the object is $x(t)$.

Newton: $mu'' =$ sum of the forces acting on the object. These include:

the spring: $F_s = -kx$ (Hooke's Law; $k > 0$)
friction: $F_f = -bx'$ ($b \geq 0$
a possible external force: $F_e = f(t)$

Putting them all together, we get $mx'' = -kx - bx' + f(t)$, i.e.,

$$mx'' + bx' + kx = f(t)$$

and this is an equation we know how to solve!

Some special cases:

No friction ($b = 0$) , i.e., *undamped*; no external force, i.e., *unforced*). Solutions are

$$u = c_1\cos(\omega_0 t) + c_2\sin(\omega_0 t) = C\cos(\omega_0 t - \delta)$$

where $\omega_0 = \sqrt{k/m}$ = the *natural frequency* of the system, C = amplitude of the vibration, δ (= 'delay') = phase angle, *where*

$C = \sqrt{(c_1^2 + c_2^2)}$, $\tan(\delta) = c_2/c_1$

[[**you are not responsible for these formulas; they are included FYI only.**]]

$T = 2\pi/\omega_0$ = period of the vibration. Note that a stiffer spring (= larger k) gives higher frequency, shorter period. Larger m gives the opposite.

Damped unforced vibrations: solve the auxiliary equation, solutions have $b^2 - 4km$ = discriminant inside of the square root, and so the solutions depend on the *sign* of the discriminant.

$b^2 > 4km$ (overdamped); fundamental solutions are $e^{r_1 t}, e^{r_2 t}$, $r_1, r_2 < 0$ (roots are negative because $m, b, k > 0$)

$b^2 = 4km$ (critically damped); fundamental solutions are e^{rt}, te^{rt}, $r < 0$

$b^2 < 4km$ (underdamped); fundamental solutions are $e^{rt}\cos(\omega t), e^{rt}\sin(\omega t)$, $r < 0$, $\omega = \sqrt{\omega_0^2 - (b/2m)^2}$

In each case, solutions tend to 0 as t goes to ∞. In first two cases, the solution has at most one local max or min; in the third, it continues to oscillate forever.

Forced vibrations: Focus on periodic forcing term: $f(t) = F_0\cos(\omega t)$.

Damped case: if we include friction ($b \neq 0$), then the solution turns out to be

$$x = \text{homog. soln. } + C\sin(\omega t - \delta)$$

But since $b > 0$, the homogeneous solutions will tend to 0 as $t \to \infty$; they are called the *transient solution.* (Basically, they just allow us to solve any initial value problem. We can then conclude that any energy given to the system is dissipated over time; leaving only the energy imparted by the forcing term to drive the system along.) The other term is called the *forced response*, or *steady-state solution.*

Undamped: when $\omega = \omega_0$, our forcing term is a solution to the homogeneous equation, so the general solution, instead, is

$$x = C_1\sin(\omega_0 t - \delta_1) + C_2 t\sin(\omega_0 t - \delta_2)$$

In this case, as t goes to ∞, the amplitude of the second oscillation goes to ∞; the solution, essentially, resonates with the forcing term. (Basically, you are 'feeding' the system at it's natural frequency.) This illustrates the phenomenon of *resonance.*

[[**And a little extra, FYI...**]]

If $\omega \neq \omega_0$, then (using undetermined coefficients) the solution is

$$x = C\cos(\omega_0 t - \delta) + \frac{F_0}{m(\omega_0^2 - \omega^2)}\cos(\omega t)$$

This is the sum of two vibrations with different frequencies.

In the special case $x(0) = 0, x'(0) = 0$ (starting at rest), we can further simplify:

$$x = \frac{2F_0}{m(\omega_0^2 - \omega^2)}\sin(\frac{\omega_0 - \omega}{2}t)\sin(\frac{\omega_0 + \omega}{2}t)$$

When ω is close to ω_0, this illustrates the concept of *beats*; we have a high frequency vibration (the second sine) with *amplitude* a low frequency vibration (the first sine). the mass essentially vibrates rapidly between to sine curves.