Math 310 Some practice problems for Exam 1

Prove by induction:

$$3(7^n) + 17(2^n) \text{ is divisible by 5, for all n} \geq 0.$$

Use the Euclidean algorithm to determine the g.c.d. of 432 and 831. Then reverse the calculations to write the g.c.d. as a linear combination of the two.

Show that the equation $3x^2 - y^3 = 176$ has no solutions with x and y integers, by considering the equation in $\mathbb{Z}_9$.

Show that if n is **odd**, then the g.c.d. of n and n+8 is always 1. (Hint: show that any k>1 that divides n can't divide n+8.)

Show that $a^2 \equiv 16 \pmod{10}$ implies $a^2 \equiv 16 \pmod{20}$.
(Hint: show that 10|(a−4)(a+4) implies 5 divides one of the factors and 2 divides **both** of them (a−4 is even if and only if a+4 is even!).)

Use the Euclidean algorithm to find $d = (217, 133)$ and find integers x, y such that $d = 217x + 133y$.

Find the least non-negative residue of $3^{116} \pmod{29}$.

Let p be a prime integer and suppose for some $a \in \mathbb{Z}_p$ that $a^2 = a$. Prove that $a = [0]_p$ or $a = [1]_p$ in $\mathbb{Z}_p$. Also, give an example to show that this can be false if p is not a prime.

Prove by mathematical induction that 3 is a divisor of $2^{2n+1} + 1$ for every positive integer n.

Prove that $\sqrt{15}$ is irrational.

Find the smallest positive integer in the set $\{10u + 15v : u, v \in \mathbb{Z}\}$. Write a sentence or two justifying your answer.

Prove that if a, b and c are integers such that $a|b$ and $a|(b + c)$ then $a|c$.

What is the remainder when one divides $(127)(244)(14)(-45)$ by 13? (You *don't* need to actually perform long division.)

If p is a positive prime number and $p|a^2$, prove that $p|a$. (Be sure to state completely any definition or theorem you use.)

Prove: If $[a] = [1]$ in $\mathbb{Z}_n$, then $(a, n) = 1$.