Math 310 Some practice problems for Exam 2

1. Suppose $f : \mathbb{Z} \to \mathbb{Z}_6$ is the function $f(x) = [x]_6$.

1.a. Show that f is a homomorphism.

1.b. Show that f is surjective.

1.c. Show that f is not injective.

2. Show that the rings $\mathbb{Z}_4$ and $\mathbb{Z}_2 \times \mathbb{Z}_2$ are not isomorphic.

3. Let $D : \mathbb{R}[x] \to \mathbb{R}[x]$ be the derivative map given by $D(a_0 + a_1x + a_2x^2 + \cdots a_nx^n) = a_1 + 2a_2x + \cdots + na_nx^{n-1}$. Show that D is not a homomorphism. (Hint: Compute $D(x^2)$.)

4. Suppose that R and S are rings, R' is a subring of R, and S' is a subring of S. Show that $R' \times S'$ is a subring of $R \times S$.

5. Show that $\mathbb{Z}_6 \times \mathbb{Z}_5 \cong \mathbb{Z}_{10} \times \mathbb{Z}_3$.

6. Let R be a ring with identity. An element $e \in R$ is called *idempotent* if $e^2 = e$. The elements 0 and 1 are called the *trivial* idempotents of R. All other idempotents (if any exist) are called *nontrivial* idempotents

Let R, S be rings with identity with $R \neq 0$, $S \neq 0$ (that is, neither R nor S is the "stupid" ring). Show that $R \times S$ always has nontrivial idempotents.

7. Find the solutions to the system of congruences

$x \equiv 3 \pmod 5$
$x \equiv 1 \pmod 6$
$x \equiv 2 \pmod{11}$

8. Let f:$\mathbb{Z}_8 \to \mathbb{Z}_{12}$ be given by f($[x]_8$) = $[3x]_{12}$.

(a): Show that f is a **well-defined** homomorphism of **groups** (under addition).

(b): Show (by example) that f is **neither** injective **nor** surjective.

(c): Is f a homomorphism of **rings**? Show why or why not.

9. Let G be an **abelian** group, with identity element e.

(a): Show that if $a, b \in G$, and $a^n = e$, $b^m = e$ for some $n, m \in \mathbb{N}$, then $(ab)^{nm} = e$.

(b): Show that $H = \{a \in G : a^k = e \text{ for some } k \geq 1\}$ is a **subgroup** of G.

10. Show that if G is a group, and $H, K \subseteq G$ are subgroups of G, then $H \cap K = \{g \in G : g \in H \text{ and } g \in K\}$ is also a subgroup of G.