Math 310 Homework #1 Solutions

1. Show $\sum_{i=1}^{n} i^3 = \left(\frac{n(n+1)}{2}\right)^2 = \frac{1}{4}n^2(n^2+2n+1) = \frac{1}{4}n^4+\frac{1}{2}n^3+\frac{1}{4}n^2$

By induction:

(1) $n=1$ $\quad \sum_{i=1}^{1} i^3 = 1 \quad \left(\frac{1(1+1)}{2}\right)^2 = \left(\frac{2}{2}\right)^2 = 1$ ✓

(2) Suppose $\sum_{i=1}^{n} i^3 = \left(\frac{n(n+1)}{2}\right)^2$. Then

$$\sum_{i=1}^{n+1} i^3 = (n+1)^3 + \sum_{i=1}^{n} i^3 = (n+1)^3 + \left(\frac{1}{4}n^4+\frac{1}{2}n^3+\frac{1}{4}n^2\right)$$
$$= n^3+3n^2+3n+1 + \frac{1}{4}n^4+\frac{1}{2}n^3 + \frac{1}{4}n^2$$
$$= \frac{1}{4}n^4 + \frac{3}{2}n^3 + \frac{13}{4}n^2 + 3n+1 .$$

But! $\left(\frac{(n+1)((n+1)+1)}{2}\right)^2 = \left(\frac{(n+1)(n+2)}{2}\right)^2 = \frac{1}{4}(n^2+3n+2)^2$

$$= \frac{1}{4}(n^4+3n^3+2n^2+3n^3+9n^2+6n+2n^2+6n+4) = \frac{1}{4}(n^4+6n^3+13n^2+12n+4)$$
$$= \frac{1}{4}n^4+\frac{3}{2}n^3+\frac{13}{4}n^2+3n+1$$

So $\sum_{i=1}^{n+1} i^3 = \left(\frac{(n+1)((n+1)+1)}{2}\right)^2$ ✓

So by P.M.I., $\sum_{i=1}^{n} i^3 = \left(\frac{n(n+1)}{2}\right)^2$ for all $n \geq 1$.

2. Show $4\cdot 5^n + 7\cdot 27^n$ is a multiple of 11 for all $n \geq 0$

By induction:

(1) $n=0$ $\quad 4\cdot 5^0 + 7\cdot 27^0 = 4\cdot 1 + 7\cdot 1 = 4+7 = 11 = 11\cdot 1$ ✓

(2) Suppose $4\cdot 5^n + 7\cdot 27^n = 11k$ for some integer k.

Then $4\cdot 5^{n+1} + 7\cdot 27^{n+1} = (4\cdot 5^n)\cdot 5 + (7\cdot 27^n)\cdot 27$

$$= 5\cdot(4\cdot 5^n + 7\cdot 27^n) + (27-5)\cdot(7\cdot 27^n)$$
$$= 5\cdot(11k) + 22\cdot(7\cdot 27^n) = 11\cdot(5k + 2\cdot 7\cdot 27^n)$$

is a multiple of 11.

So, by P.M.I., $4\cdot 5^n + 7\cdot 27^n$ is a multiple of n for all $n \geq 0$.

3. Show $55\cdot 44^n - 6\cdot 23^n$ is a multiple of 7 for all $n \geq 0$.

By induction:

(1) $n=0$ $\quad 55\cdot 44^0 - 6\cdot 23^0 = 55 - 6 = 49 = 7\cdot 7$ ✓

(2) If $55\cdot 44^n - 6\cdot 23^n = 7k$, then

$$55\cdot 44^{n+1} - 6\cdot 23^{n+1} = 44\cdot(55\cdot 44^n) - 23\cdot(6\cdot 23^n)$$
$$= 23\,(55\cdot 44^n - 6\cdot 23^n) + (44-23)\cdot(6\cdot 23^n)$$
$$= 23\cdot(7k) + (21)\cdot(6\cdot 23^n) = 7\cdot(23k + 3\cdot 6\cdot 23^n)$$

is also a multiple of 7. ✓

So by P.M.I., $55\cdot 44^n - 6\cdot 23^n$ is a multiple of 7, for all $n \geq 0$.

4. For every <u>odd</u> $m \geq 1$, $4^m + 5^m$ is a multiple of 9.

m is odd means $m = 2k+1$. $m \geq 1$ means $k \geq 0$. So we want: for all $k \geq 0$ $4^{2k+1} + 5^{2k+1}$ is a multiple of 9.

Prove by induction!

(1) $k=0$ $\quad 4^{2\cdot 0+1} + 5^{2\cdot 0+1} = 4^1 + 5^1 = 4 + 5 = 9 = 9\cdot 1$ ✓

(2) If $4^{2k+1} + 5^{2k+1} = 9\cdot \ell$, then

$$4^{2(k+1)+1} + 5^{2(k+1)+1} = 4^{(2k+1)+2} + 5^{(2k+1)+2}$$
$$= 16(4^{2k+1}) + 25(5^{2k+1}) = 16(4^{2k+1} + 5^{2k+1}) + (25-16)(5^{2k+1})$$
$$= 16(9\ell) + 9(5^{2k+1}) = 9(16\ell + 5^{2k+1})$$

is a multiple of 9. ✓

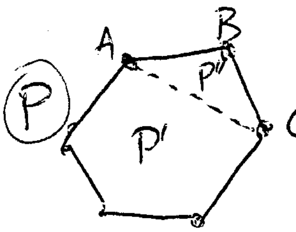
So, by P.M.I., $4^m + 5^m$ is a multiple of 9 for all <u>odd</u> $m \geq 1$.

5. For any convex polygon with n sides, the sum of the interior angles is $(n-2)\pi$.

By complete induction:

(1) smallest n making a polygon is $n=3$ (triangle). The sum of the angles of a triangle is $\pi = (3-2)\pi$ (from high school geometry?). ✓

(2) Suppose that every convex polygon with $3 \le k < n$ sides has sum of interior angles $= (k-2)\pi$. Then for a convex polygon with n sides, ~~xxxxx~~ with 3 adjacent vertices A, B, C,

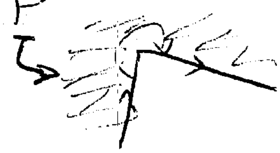


draw the line segment AC. This cuts the convex polygon (call it P) ~~∃~~ into two convex polygons P' and P'', one having $(n-1)$ sides (P', say), and one having $3 < n$ sides (P'', say). By our inductive hypothesis, the sum of the interior angles of P' is $((n-1)-2)\pi = (n-3)\pi$, and the sum of the interior angles of P'' is π. But! from the picture, the interior angles of P' and P'' together add up to the interior angles of P. So the sum of the interior angles of P is

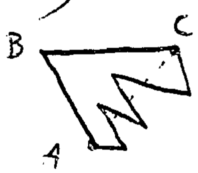
$$(n-3)\pi + \pi = (n-2)\pi. \checkmark$$

So by complete induction, the sum of the interior angles of a convex! polygon with n sides is $(n-2)\pi$.

[FYI: This result is also true for polygons that aren't convex, but you need to be much more careful: you need to allow "reflex" angles ($>\pi$) inside P

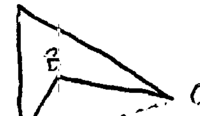


, and you to worry that the line segment AC hits P:



, ∃

and you need to worry that AC is outside of P:



.]

6. $S \subseteq \mathbb{Z}$ so that for some $N \in \mathbb{Z}$, $s \leq N$ for all $s \in S$. Then S has a largest element, i.e. $\exists\ s_0 \in S$ so that $s \leq s_0$ for all $s_0 \in S$.

<u>Note</u>: <u>need</u> $S \neq \emptyset$, otherwise "$\exists\ s_0 \in S$" (forget the rest...) is impossible.

If $S \neq \emptyset$ and N is as above, set $A = \{N - s, \text{ where } s \in S\}$. Then $A \neq \emptyset$ (there's at least one $N-s$), and since $s \leq N$ for all $s \in S$, $N - s \geq 0$ for all $s \in S$, so $a \geq 0$ for all $a \in A$, i.e. $A \subseteq \mathbb{N}$. Then by well-ordering, A has a <u>smallest</u> element a_0, i.e. $a_0 \in A$ and $a_0 \leq a$ for all $a \in A$. But then

$a_0 = N - s_0$ for some $s_0 \in S$. Then for any $s \in S$,

$N - s = a \in A$, so $N - s_0 = a_0 \leq a = N - s$, so $N - s_0 \leq N - s$,

so $N - s_0 + (s_0 + s) = N + s \leq N - s + (s_0 + s) = N + s_0$, so

$N + s - N = s \leq N + s_0 - N = s_0$, i.e. $s \leq s_0$ for all $s \in S$.

So s_0 is the largest element of S. ///