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Solutions

Name:

Math 314 Matrix Theory

Exam 2

Show all work. How you get your answer is just as important, if not more important, than the answer itself. If you think it, write it!

1. (20 pts.) Find the determinant of the matrix

$$A = \begin{pmatrix} 1 & 2 & 1 & -1 \\ 3 & -1 & 2 & 3 \\ 5 & 4 & 2 & 1 \\ 2 & 4 & 2 & -3 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 2 & 1 & -1 \\ 3 & -1 & 2 & 3 \\ 5 & 4 & 2 & 1 \\ 2 & 4 & 2 & -3 \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & 2 & 1 & -1 \\ 0 & -7 & -1 & 6 \\ 0 & -6 & -3 & 6 \\ 0 & 0 & 0 & -1 \end{pmatrix} \xrightarrow{-1} \begin{pmatrix} 1 & 2 & 1 & -1 \\ 0 & -6 & -3 & 6 \\ 0 & -7 & -1 & 6 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

$$\xrightarrow{-\frac{1}{6}} \begin{pmatrix} 1 & 2 & 1 & -1 \\ 0 & 1 & 1/2 & -1 \\ 0 & -7 & -1 & 6 \\ 0 & 0 & 0 & -1 \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & 2 & 1 & -1 \\ 0 & 1 & 1/2 & -1 \\ 0 & 0 & 5/2 & -1 \\ 0 & 0 & 0 & -1 \end{pmatrix} = R$$

$$de(R) = (1)(1)(\tfrac{5}{2})(-1) = (-1)(-\tfrac{1}{6})\,det(A) \text{ , so}$$

$$det(A) = (-1)(-6)(1)(1)(\tfrac{5}{2})(-1) = 6(\tfrac{-5}{2}) = (\tfrac{-30}{2}) = \boxed{-15}$$

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2. (20 pts.) For the vector space $\mathcal{P}_3$ of polynomials of degree less than or equal to 3, let $T : \mathcal{P}_3 \to \mathbf{R}$ be the function

$$T(p) = p(2) + p(3) \ .$$

Show that T is a *linear transformation*, and find numbers a, b, and c so that

$$T(x + a) = T(x^2 + b) = T(x^3 + c) = 0 \ .$$

We want: $T(p+q) = T(p) + T(q)$, $T(cp) = cT(p)$
for $c \in \mathbb{R}$, $p, q \in \mathcal{P}_3$.

But

$$T(p+q) = (p+q)(2) + (p+q)(3)$$
$$= (p(2) + q(2)) + (p(3) + q(3)) = (p(2) + p(3)) + (q(2) + q(3))$$
$$= T(p) + T(q) \quad ✓$$

$$T(cp) = (cp)(2) + (cp)(3) = c(p(2)) + c(p(3))$$
$$= c(p(2) + p(3)) = cT(p) \quad ✓$$

So: T is a linear transformation.

$$T(x+a) = (2+a) + (3+a) = 2a + 5 = 0 \text{ for } a = \frac{-5}{2}$$
$$T(x^2+b) = (4+b) + (9+b) = 2b + 13 = 0 \text{ for } b = -13/2$$
$$T(x^3+c) = (8+c) + (27+c) = 2c + 35 = 0 \text{ for } c = -35/2$$

So $T(x - \frac{5}{2}) = T(x^2 - \frac{13}{2}) = T(x^3 - \frac{35}{2}) = 0$. ///

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3. (25 pts.) Find bases for the column, row, and nullspaces of the matrix

$$B = \begin{pmatrix} 1 & 2 & 1 & -1 \\ 3 & -1 & 2 & 3 \\ -3 & 8 & -1 & -9 \\ 5 & 3 & 4 & 1 \end{pmatrix}.$$

Raw reduce!

$$\begin{pmatrix} 1 & 2 & 1 & -1 \\ 3 & -1 & 2 & 3 \\ -3 & 8 & -1 & -9 \\ 5 & 3 & 4 & 1 \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & 2 & 1 & -1 \\ 0 & -7 & -1 & 6 \\ 0 & 14 & 2 & -12 \\ 0 & -7 & -1 & 6 \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & 2 & 1 & -1 \\ 0 & -7 & -1 & 6 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\longrightarrow \begin{pmatrix} 1 & 2 & 1 & -1 \\ 0 & 1 & 1/7 & -6/7 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & 0 & 5/7 & 5/7 \\ 0 & 1 & 1/7 & -6/7 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$\uparrow\ \uparrow$ pivots $\qquad \uparrow\ \uparrow$ free.

$$x + 5/7 z + 5/7 w = 0$$
$$y + 1/7 z - 6/7 w = 0$$
$$x = -5/7 z - 5/7 w$$
$$y = -1/7 z + 6/7 w$$

$$\begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix} = \begin{pmatrix} -5/7 z - 5/7 w \\ -1/7 z + 6/7 w \\ z \\ w \end{pmatrix}$$

So: $\begin{pmatrix} 1 \\ 3 \\ -3 \\ 5 \end{pmatrix}, \begin{pmatrix} 2 \\ -1 \\ 8 \\ 3 \end{pmatrix}$ = basis for Col(B)

$\begin{pmatrix} 1 \\ 0 \\ 5/7 \\ 5/7 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 1/7 \\ -6/7 \end{pmatrix}$ = basis for Raw(B)

$\begin{pmatrix} -5/7 \\ -1/7 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -5/7 \\ 6/7 \\ 0 \\ 1 \end{pmatrix}$ = basis for Nul(B)

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4. (20 pts.) Show that the collection of vectors $W = \{(a\ b\ c)^T \in \mathbf{R}^3\ :\ 3a - 2b + c = 0\}$ is a *subspace* of $\mathbf{R}^3$, and find a *basis* for W.

Need: $\vec{v},\vec{w} \in W \Longrightarrow \vec{v}+\vec{w} \in W$

$\vec{v} \in W,\ c \in \mathbb{R} \Longrightarrow c\vec{v} \in W$

$\vec{v} = \begin{pmatrix} a\\ b\\ c\end{pmatrix}$, $\vec{w} = \begin{pmatrix} x\\ y\\ z\end{pmatrix}$, so $3a-2b+c=0$, $3x-2y+z=0$, then

$\vec{v}+\vec{w} = \begin{pmatrix} a+x\\ b+y\\ c+z\end{pmatrix}$, and $3(a+x)-2(b+y)+(c+z)$

$= (3a-2b+c) + (3x-2y+z) = 0+0 = 0$

so $\vec{v}+\vec{w} \in W$. ✓

$k\vec{v} = \begin{pmatrix} ka\\ kb\\ kc\end{pmatrix}$, and $3(ka)-2(kb)+(kc)$

$= k(3a-2b+c) = k(0) = 0$

so $k\vec{v} \in W$ ✓

so W is a subspace.

W looks like a nullspace!

$(3\ -2\ 1)\begin{pmatrix} a\\ b\\ c\end{pmatrix} = 0$, so $W = \text{Nul}(3\ -2\ 1)$.

Basis: row reduce! $(3\ -2\ 1) \longrightarrow (1\ -\frac{2}{3}\ \frac{1}{3})$

$x - \frac{2}{3}y + \frac{1}{3}z = 0$

$x = \frac{2}{3}y - \frac{1}{3}z$

$\begin{pmatrix} x\\ y\\ z\end{pmatrix} = \begin{pmatrix} \frac{2}{3}y - \frac{1}{3}z\\ y\\ z\end{pmatrix} = y\begin{pmatrix} \frac{2}{3}\\ 1\\ 0\end{pmatrix} + z\begin{pmatrix} -\frac{1}{3}\\ 0\\ 1\end{pmatrix}$

Basis: $\begin{pmatrix} \frac{2}{3}\\ 1\\ 0\end{pmatrix}, \begin{pmatrix} -\frac{1}{3}\\ 0\\ 1\end{pmatrix}$

5. (15 pts.) If a 5×8 matrix C has rank equal to 4, what is the dimension of its nullspace (and why does it have that value?)?

$4 = \text{rank}(C) = \dim(\text{Col}(C)) =$ # of pivots in (R)REF of C. C has 8 columns, so with 4 pivots, this means it has 4 free variables in (R)REF.

But $\dim(\text{Nul}(C)) =$ # of free variables in (R)REF,

so $\dim(\text{Nul}(C)) = \boxed{4} = 8-4$.