

### Math 314/814 Exam 1 Practice Problems Solutions (part)

1. (20 pts.) Use row reduction to find a solution to the following system of linear equations:

$$\begin{aligned} -x & & + 2z & = 3 \\ x & - y & - 3z & = 3 \\ 2x & + 3y & - 3z & = 6 \end{aligned}$$

We can express this as an augmented matrix and row reduce:

$$\begin{aligned} \left( \begin{array}{ccc|c} -1 & 0 & 2 & 3 \\ 1 & -1 & -3 & 3 \\ 2 & 3 & -3 & 6 \end{array} \right) &\rightarrow \left( \begin{array}{ccc|c} 1 & 0 & -2 & -3 \\ 1 & -1 & -3 & 3 \\ 2 & 3 & -3 & 6 \end{array} \right) &\rightarrow \left( \begin{array}{ccc|c} 1 & 0 & -2 & -3 \\ 0 & -1 & -1 & 6 \\ 0 & 3 & 1 & 12 \end{array} \right) &\rightarrow \\ \left( \begin{array}{ccc|c} 1 & 0 & -2 & -3 \\ 0 & 1 & 1 & -6 \\ 0 & 3 & 1 & 12 \end{array} \right) &\rightarrow \left( \begin{array}{ccc|c} 1 & 0 & -2 & -3 \\ 0 & 1 & 1 & -6 \\ 0 & 0 & -2 & 30 \end{array} \right) &\rightarrow \left( \begin{array}{ccc|c} 1 & 0 & -2 & -3 \\ 0 & 1 & 1 & -6 \\ 0 & 0 & 1 & -15 \end{array} \right) &\rightarrow \\ \left( \begin{array}{ccc|c} 1 & 0 & 0 & -33 \\ 0 & 1 & 0 & 9 \\ 0 & 0 & 1 & -15 \end{array} \right) \end{aligned}$$

so our solution is  $x = -33, y = 9, z = -15$ .

2. (20 pts.) Use row reduction to decide if the system of linear equations given by the augmented matrix:

$$(A|\mathbf{b}) = \left( \begin{array}{cccc|c} -1 & 0 & 2 & 3 & 2 \\ 3 & 2 & -4 & 1 & -2 \\ 0 & 3 & 0 & 1 & 6 \end{array} \right)$$

has a solution. If it does, does it have one or more than one solution?

We row reduce:

$$\begin{aligned} \left( \begin{array}{cccc|c} -1 & 0 & 2 & 3 & 2 \\ 3 & 2 & -4 & 1 & -2 \\ 0 & 3 & 0 & 1 & 6 \end{array} \right) &\rightarrow \left( \begin{array}{cccc|c} 1 & 0 & -2 & -3 & -2 \\ 0 & 1 & 1 & 5 & 2 \\ 0 & 3 & 0 & 1 & 6 \end{array} \right) &\rightarrow \\ \left( \begin{array}{cccc|c} 1 & 0 & -2 & -3 & -2 \\ 0 & 1 & 1 & 5 & 2 \\ 0 & 0 & -3 & -14 & 0 \end{array} \right) &\rightarrow \left( \begin{array}{cccc|c} 1 & 0 & -2 & -3 & -2 \\ 0 & 1 & 1 & 5 & 2 \\ 0 & 0 & 1 & 14/3 & 0 \end{array} \right) \end{aligned}$$

This is enough to answer the question: the system of equations is consistent - there is no row of zeros in the coefficient matrix, in REF, opposite a non-zero number - so there is a solution. More, there is a free variable - the last column in the coefficient matrix - so there are more than one solution.

3. (25 pts.) Is the vector  $\vec{b} = \begin{bmatrix} -1 \\ 3 \\ 1 \end{bmatrix}$  in the span of the vectors  $\vec{v}_1 = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$ ,  $\vec{v}_2 = \begin{bmatrix} 1 \\ 3 \\ -2 \end{bmatrix}$  ?

More generally, what equation among  $a, b, c$  must hold in order for

$\begin{bmatrix} a \\ b \\ c \end{bmatrix}$  to be in the span of  $\vec{v}_1, \vec{v}_2$  ?

The first question asks if  $\left( \begin{array}{cc|c} 1 & 1 & -1 \\ 2 & 3 & 3 \\ 3 & -2 & 1 \end{array} \right)$  has a solution. So we row reduce:

$$\left( \begin{array}{cc|c} 1 & 1 & -1 \\ 2 & 3 & 3 \\ 3 & -2 & 1 \end{array} \right) \rightarrow \left( \begin{array}{cc|c} 1 & 1 & -1 \\ 0 & 1 & 5 \\ 0 & -5 & 4 \end{array} \right) \rightarrow \left( \begin{array}{cc|c} 1 & 0 & -6 \\ 0 & 1 & 5 \\ 0 & 0 & 29 \end{array} \right)$$

This is inconsistent: the last row reads  $0 = 29$ . So there are no solutions, so  $\vec{b}$  is not in the span of  $\vec{v}_1$  and  $\vec{v}_2$ .

The second question asks when does  $\left( \begin{array}{cc|c} 1 & 1 & a \\ 2 & 3 & b \\ 3 & -2 & c \end{array} \right)$  have a solution. We row reduce:

$$\left( \begin{array}{cc|c} 1 & 1 & a \\ 2 & 3 & b \\ 3 & -2 & c \end{array} \right) \rightarrow \left( \begin{array}{cc|c} 1 & 1 & a \\ 0 & 1 & -2a+b \\ 0 & -5 & -3a+c \end{array} \right) \rightarrow \left( \begin{array}{cc|c} 1 & 0 & 3a-b \\ 0 & 1 & -2a+b \\ 0 & 0 & -13a+5b+c \end{array} \right)$$

which will have a solution exactly when  $-13a + 5b + c = 0$  (so that the system is

consistent. So  $\begin{bmatrix} a \\ b \\ c \end{bmatrix}$  will be in the span precisely when  $-13a + 5b + c = 0$ ; this equation must hold in order to be in the span.

4. (20 pts.) Use Gauss-Jordan elimination to find the inverse of the matrix  $A$ , where

$$A = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 0 & 1 \\ 3 & 5 & 8 \end{pmatrix}$$

We write down the supraugmented matrix and row reduce:

$$\begin{pmatrix} 1 & 2 & 3 & | & 1 & 0 & 0 \\ 2 & 0 & 1 & | & 0 & 1 & 0 \\ 3 & 5 & 8 & | & 0 & 0 & 1 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 2 & 3 & | & 1 & 0 & 0 \\ 0 & -4 & -5 & | & -2 & 1 & 0 \\ 0 & -1 & -1 & | & -3 & 0 & 1 \end{pmatrix} \rightarrow$$

$$\begin{pmatrix} 1 & 2 & 3 & | & 1 & 0 & 0 \\ 0 & -1 & -1 & | & -3 & 0 & 1 \\ 0 & -4 & -5 & | & -2 & 1 & 0 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 2 & 3 & | & 1 & 0 & 0 \\ 0 & 1 & 1 & | & 3 & 0 & -1 \\ 0 & -4 & -5 & | & -2 & 1 & 0 \end{pmatrix} \rightarrow$$

$$\begin{pmatrix} 1 & 0 & 1 & | & -5 & 0 & 2 \\ 0 & 1 & 1 & | & 3 & 0 & -1 \\ 0 & 0 & -1 & | & 10 & 1 & -4 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 0 & 1 & | & -5 & 0 & 2 \\ 0 & 1 & 1 & | & 3 & 0 & -1 \\ 0 & 0 & 1 & | & -10 & -1 & 4 \end{pmatrix} \rightarrow$$

$$\begin{pmatrix} 1 & 0 & 0 & | & 5 & 1 & -2 \\ 0 & 1 & 0 & | & 13 & 1 & -5 \\ 0 & 0 & 1 & | & -10 & -1 & 4 \end{pmatrix} \quad \text{so } A^{-1} = \begin{pmatrix} 5 & 1 & -2 \\ 13 & 1 & -5 \\ -10 & -1 & 4 \end{pmatrix}$$

- (b) (5 pts.) Use your answer to find the solution to the equation  $Ax = \begin{pmatrix} 2 \\ 2 \\ 1 \end{pmatrix}$

$$A\vec{x} = \begin{pmatrix} 2 \\ 2 \\ 1 \end{pmatrix} \text{ has solution } \quad \vec{x} = A^{-1} \begin{pmatrix} 2 \\ 2 \\ 1 \end{pmatrix} = \begin{pmatrix} 5 & 1 & -2 \\ 13 & 1 & -5 \\ -10 & -1 & 4 \end{pmatrix} \begin{pmatrix} 2 \\ 2 \\ 1 \end{pmatrix} = \begin{pmatrix} 10 \\ 23 \\ -18 \end{pmatrix}$$

5. (10 pts.) Let  $\mathbf{O}$  denote the  $n \times n$  matrix with all entries equal to 0.

Suppose that  $A$  and  $B$  are  $n \times n$  matrices with

$$AB = \mathbf{O},$$

but

$$B \neq \mathbf{O}.$$

Show that  $A$  **cannot** be invertible.

(Hint: suppose it *is*: what does that tell you about  $B$ ?)

If  $A$  has an inverse  $A^{-1}$ , then  $A^{-1}A = I_n$ , the  $n \times n$  identity matrix. But then  $B = I_n B = (A^{-1}A)B = A^{-1}(AB) = A^{-1}\mathbf{O} = \mathbf{O}$ , the last because multiplying a row by a column of 0s gives 0. So if  $A$  is invertible, then  $B = \mathbf{O}$ . But that's not true, so  $A$  can't be invertible.