Math 314

Exam 2 practice problems

Solutions

Solutions

Name:

Math 314 Matrix Theory

Exam 2

Show all work. How you get your answer is just as important, if not more important, than the answer itself. If you think it, write it!

1. (20 pts.) Find, using any method (other than psychic powers), the determinant of the matrix

$$A = \begin{pmatrix} 1 & 2 & 1 \\ 0 & -1 & 3 \\ 4 & 4 & 2 \end{pmatrix}$$

Is this matrix invertible?

$$\longrightarrow \begin{pmatrix} 1 & 2 & 1 \\ 0 & -1 & 3 \\ 0 & -4 & -2 \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & 2 & 1 \\ 0 & -1 & 3 \\ 0 & 0 & -14 \end{pmatrix}$$

$$\det(A) = 1 \cdot (-1)(-14) = 14$$

$\det(A) \neq 0$ so A is invertible

or

$$\det(A) = 1 \begin{vmatrix} -1 & 3 \\ 4 & 2 \end{vmatrix} - 0 \begin{vmatrix} 2 & 1 \\ 4 & 2 \end{vmatrix} + 4 \begin{vmatrix} 2 & 1 \\ -1 & 3 \end{vmatrix}$$

$$= 1(-14) + 4(7) = \underline{\underline{14}}.$$

2. (15 pts.) Explain why the set of vectors

$$W = \{(x, y, z) \mid x + y + 2z = 1\}$$

is **not** a subspace of $\mathbf{R}^3$.

How many reasons do you want?

$0 + 0 + 2(0) = 0 \neq 1$ so $(0,0,0) \notin W$
so it can't be a subspace.

$\overset{u}{(1,0,0)}, \overset{v}{(0,1,0)} \in W$ but:

(a) $u+v = (1,1,0)$ has $1+1+2(0) = 2 \neq 1$ so
$u+v \notin W$, so it can't be...

(b) $2 \cdot u = (2,0,0)$ has $2+0+2(0) = 2 \neq 1$
so $2u \notin W$ so it can't be...

$-u = (-1,0,0)$ has $(-1)+0+2(0) = -1 \neq 1$ so it
can't be....

Any one answer will do!

3. (25 pts.) Show that the system of equations $Ax = b$, where

$$A = \begin{pmatrix} -1 & 1 \\ 1 & -2 \\ 0 & 1 \end{pmatrix} \text{ and } b = \begin{pmatrix} 2 \\ 2 \\ -1 \end{pmatrix}$$

is **not** consistent. Find the least squares solution to this sytem, i.e., the value of Ax closest to b.

$$\left(\begin{array}{cc|c} -1 & 1 & 2 \\ 1 & -2 & 2 \\ 0 & 1 & -1 \end{array}\right) \to \left(\begin{array}{cc|c} 1 & -1 & -2 \\ 0 & -1 & 4 \\ 0 & 1 & -1 \end{array}\right) \to \left(\begin{array}{cc|c} 1 & 0 & -3 \\ 0 & 1 & -4 \\ 0 & 0 & 3 \end{array}\right) \leftarrow \text{inconsistent}$$

Last line says "$0 = 3$" ...

$$A^TA = \begin{pmatrix} -1 & 1 & 0 \\ 1 & -2 & 1 \end{pmatrix}\begin{pmatrix} -1 & 1 \\ 1 & -2 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 2 & -3 \\ -3 & 6 \end{pmatrix}$$

$$\det(A^TA) = 12 - 9 = 3 \qquad (A^TA)^{-1} = \frac{1}{3}\begin{pmatrix} 6 & 3 \\ 3 & 2 \end{pmatrix}$$

$$A^T\vec{b} = \begin{pmatrix} -1 & 1 & 0 \\ 1 & -2 & 1 \end{pmatrix}\begin{pmatrix} 2 \\ 2 \\ -1 \end{pmatrix} = \begin{pmatrix} 0 \\ -3 \end{pmatrix}$$

$$A^TA\vec{x} = A^T\vec{b} \qquad \vec{x} = (A^TA)^{-1}(A^T\vec{b})$$

$$= \frac{1}{3}\begin{pmatrix} 6 & 3 \\ 3 & 2 \end{pmatrix}\begin{pmatrix} 0 \\ -3 \end{pmatrix} = \frac{1}{3}\begin{pmatrix} -9 \\ -6 \end{pmatrix} = \begin{pmatrix} -3 \\ -2 \end{pmatrix}$$

$$A\vec{x} = \begin{pmatrix} -1 & 1 \\ 1 & -2 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} -3 \\ -2 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ -2 \end{pmatrix} = \text{value of } A\vec{x} \text{ closest to } \vec{b}.$$

4.(20 pts.) For the matrix

$$A = \begin{pmatrix} 1 & 2 & 1 & 0 \\ 2 & 1 & 3 & 2 \\ 1 & 1 & 1 & 1 \end{pmatrix}$$

find bases for, and the dimensions of, the row, column, and null spaces of A.

$$A \longrightarrow \begin{pmatrix} 1 & 2 & 1 & 0 \\ 0 & -3 & 1 & 2 \\ 0 & -1 & 0 & 1 \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & 2 & 1 & 0 \\ 0 & 1 & 0 & -1 \\ 0 & -3 & 1 & 2 \end{pmatrix}$$

$$\longrightarrow \begin{pmatrix} 1 & 2 & 1 & 0 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & -1 \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & 0 & 1 & 2 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & -1 \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & 0 & 0 & 3 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & -1 \end{pmatrix}$$

$\mathcal{R}(A)$: basis $= \begin{pmatrix} 1 \\ 0 \\ 0 \\ 3 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \\ -1 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \\ -1 \end{pmatrix}$ $\quad$ dim $= \underline{3}$

$\mathcal{C}(A)$: basis $= \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}, \begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 3 \\ 1 \end{pmatrix}$ $\quad$ dim $= \underline{3}$

$\mathcal{N}(A)$:

$$\begin{aligned} x_1 + 3x_4 &= 0 \\ x_2 - x_4 &= 0 \\ x_3 - x_4 &= 0 \end{aligned} \qquad \begin{aligned} x_1 &= -3x_4 \\ x_2 &= x_4 \\ x_3 &= x_4 \end{aligned} \qquad \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = x_4 \underbrace{\begin{pmatrix} -3 \\ 1 \\ 1 \\ 1 \end{pmatrix}}_{\text{basis}} \qquad \text{dim} = \underline{1}$$

5. (20 pts.) Find **all** of the solutions to the equation $Ax = b$, where

$$A = \begin{pmatrix} 1 & 2 & 2 & 1 \\ 2 & 4 & 3 & 3 \\ 1 & 2 & 1 & 2 \end{pmatrix} \text{ and } b = \begin{pmatrix} -2 \\ -2 \\ 0 \end{pmatrix}$$

$$\left(\begin{array}{cccc|c} 1 & 2 & 2 & 1 & -2 \\ 2 & 4 & 3 & 3 & -2 \\ 1 & 2 & 1 & 2 & 0 \end{array}\right) \longrightarrow \left(\begin{array}{cccc|c} 1 & 2 & 2 & 1 & -2 \\ 0 & 0 & -1 & 1 & 2 \\ 0 & 0 & -1 & +1 & 2 \end{array}\right)$$

$$\longrightarrow \left(\begin{array}{cccc|c} 1 & 2 & 2 & 1 & -2 \\ 0 & 0 & 1 & -1 & -2 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right) \longrightarrow \left(\begin{array}{cccc|c} 1 & 2 & 0 & 3 & 2 \\ 0 & 0 & 1 & -1 & -2 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right)$$

$x_1 + 2x_2 + 3x_4 = 2$ $\qquad$ $x_1 = 2 - 2x_2 - 3x_4$

$x_3 - x_4 = -2$ $\qquad$ $x_3 = -2 + x_4$

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 2 \\ 0 \\ -2 \\ 0 \end{pmatrix} + x_2 \begin{pmatrix} -2 \\ 1 \\ 0 \\ 0 \end{pmatrix} + x_4 \begin{pmatrix} -3 \\ 0 \\ 1 \\ 1 \end{pmatrix}$$

Solutions

Name:

M314 Matrix Theory
Exam 2

Exams provide you the student with an opportunity to demonstrate your understanding of the techniques presented in the course. So:

Show all work. The steps you take to your answer are just as important, if not more important, than the answer itself. If you think it, write it!

1. (20 pts.) Find, using any method (other than psychic powers), the determinant of the matrix

$$A = \begin{pmatrix} -1 & 0 & 1 & 2 \\ 3 & -3 & 1 & 1 \\ 0 & 2 & 0 & 3 \\ 1 & -1 & 1 & 1 \end{pmatrix}$$

Is this matrix invertible?

$$|A| = (-1)\begin{vmatrix} 1 & 0 & -1 & -2 \\ 0 & -3 & 4 & 7 \\ 0 & 2 & 0 & 3 \\ 0 & -1 & 2 & 3 \end{vmatrix} \Rightarrow (-1)^3 \begin{vmatrix} 1 & 0 & -1 & -2 \\ 0 & 1 & -2 & -3 \\ 0 & 2 & 0 & 3 \\ 0 & -3 & 4 & 7 \end{vmatrix} = (-1)^3 \begin{vmatrix} 1 & 0 & -1 & -2 \\ 0 & 1 & -2 & -3 \\ 0 & 0 & 4 & 9 \\ 0 & 0 & -2 & -2 \end{vmatrix}$$

$$= (-1)^3(-1)(-2)\begin{vmatrix} 1 & 0 & -1 & -2 \\ 0 & 1 & -2 & -3 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 4 & 9 \end{vmatrix} = (-1)^3(-1)(-2)\begin{vmatrix} 1 & 0 & -1 & -2 \\ 0 & 1 & -2 & -3 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 5 \end{vmatrix}$$

$$= (-2)(1)(1)(1)(5) = -10 \neq 0 \quad \therefore A \text{ is invertible.}$$

or (expand on 3rd row)

$$\det|A| = 0\begin{vmatrix} 0 & 1 & 2 \\ -3 & 1 & 1 \\ -1 & 1 & 1 \end{vmatrix} - 2\begin{vmatrix} -1 & 1 & 2 \\ 3 & 1 & 1 \\ 1 & 1 & 1 \end{vmatrix} + 0\begin{vmatrix} -1 & 0 & 2 \\ 3 & -3 & 1 \\ 1 & 1 & 1 \end{vmatrix} - 3\begin{vmatrix} -1 & 0 & 1 \\ 3 & -3 & 1 \\ 1 & -1 & 1 \end{vmatrix}$$

$$= -2\left(-1\begin{vmatrix} 1 & 1 \\ 1 & 1 \end{vmatrix} - 1\begin{vmatrix} 3 & 1 \\ 1 & 1 \end{vmatrix} + 2\begin{vmatrix} 3 & 1 \\ 1 & 1 \end{vmatrix}\right) - 3\left(-1\begin{vmatrix} -3 & 1 \\ -1 & 1 \end{vmatrix} - 0\begin{vmatrix} 3 & 1 \\ 1 & 1 \end{vmatrix} + 1\begin{vmatrix} 3 & -3 \\ -1 & 1 \end{vmatrix}\right)$$

$$= -2(0 - 2 + 4) - 3(2 - 0 + 0)$$

$$= -4 - 6 = -10$$

or [expand on any other row or column...]

5. A friend of yours runs up to you and says 'Look I've found these three vectors v_1, v_2, v_3 in $\mathbf{R}^2$ that are linearly independent!' Explain how you know, without even looking at the vectors, that your friend is wrong (again).

3 vectors in $\mathbb{R}^2$ can't be linearly independent, because if we write them as columns of a matrix and row reduce $(v_1 v_2 v_3) = A \longrightarrow R$

R ~~can~~ has pivots in different rows, so has at most 2 pivots. Since R has 3 columns, it therefore has a free variable, so $A\vec{x} = \vec{0}$ has a non-$\vec{0}$ solution. This gives a non-trivial linear combination

$av_1 + bv_2 + cv_3 = \vec{0}$, so the vectors are linearly dependent!

3. The system of equations

$$\left(\begin{array}{cccc|cccc} 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 \\ 2 & 2 & 3 & 4 & 0 & 1 & 0 & 0 \\ 3 & 3 & -1 & -6 & 0 & 0 & 1 & 0 \\ 1 & 1 & 2 & 3 & 0 & 0 & 0 & 1 \end{array}\right) \text{ row-reduces to } \left(\begin{array}{cccc|cccc} 1 & 1 & 0 & 0 & 14 & -5 & -1 & 0 \\ 0 & 0 & 1 & 0 & -24 & 9 & 2 & 0 \\ 0 & 0 & 0 & 1 & 11 & -4 & -1 & 0 \\ 0 & 0 & 0 & 0 & 1 & -1 & 0 & 1 \end{array}\right).$$

$$\begin{array}{cccc} A & I & R & Q \end{array}$$

If we call the left-hand side of the first pair of matrices A, use this row-reduction information to find the dimensions and bases for the subspaces Row(A), Nul(A), and Row(A^T).

(5 pts. for each subspace.)

Row(A) has basis (the transposes of) the non-0 rows of R, so $\begin{pmatrix}1\\1\\0\\0\end{pmatrix}, \begin{pmatrix}0\\0\\1\\0\end{pmatrix}, \begin{pmatrix}0\\0\\0\\1\end{pmatrix}$ are a basis for Row(A)

R has one free variable, so Nul(A) has one basis vector.

y is <u>free</u>.

$x+y=0$, $z=0$, $w=0$ gives $x=-y$, $y=y$, $z=0$, $w=0$ $\quad \vec{x} = y\begin{pmatrix}-1\\1\\0\\0\end{pmatrix}$ so

$\begin{pmatrix}-1\\1\\0\\0\end{pmatrix}$ is a basis for Nul(A)

Row(A^T) = Col(A), and Col(A) has basis the <u>pivot</u> columns of A. Pivots are in columns 1, 2, and 4, so $\begin{pmatrix}1\\2\\3\\1\end{pmatrix}, \begin{pmatrix}1\\3\\-1\\2\end{pmatrix}, \begin{pmatrix}1\\4\\-6\\3\end{pmatrix}$ is a basis for Row(A^T).

3. Do the vectors $\begin{pmatrix}1\\-1\\3\end{pmatrix}$, $\begin{pmatrix}2\\0\\2\end{pmatrix}$, $\begin{pmatrix}-1\\1\\1\end{pmatrix}$, and $\begin{pmatrix}1\\3\\5\end{pmatrix}$ span $\mathbf{R}^3$?

Are they linearly independent?
Can you find a subset of this collection of vectors which forms a basis for R^3?
(10 pts. for spanning, 10 pts. for lin indep, 5 pts. for basis.)

Both of the first 2 questions can be answered by row-reducing

$$\begin{pmatrix}1 & 2 & -1 & 1\\ -1 & 0 & 1 & 3\\ 3 & 2 & 1 & 5\end{pmatrix} \longrightarrow \begin{pmatrix}1 & 2 & -1 & 1\\ 0 & 2 & 0 & 4\\ 0 & -4 & 4 & 2\end{pmatrix}$$

$$\longrightarrow \begin{pmatrix}1 & 2 & -1 & 1\\ 0 & 1 & 0 & 2\\ 0 & -4 & 4 & 2\end{pmatrix} \longrightarrow \begin{pmatrix}1 & 2 & -1 & 1\\ 0 & 1 & 0 & 2\\ 0 & 0 & 4 & 10\end{pmatrix} \quad \text{REF}.$$

we have 3 pivots, so we have a pivot in every row, so they span $\mathbb{R}^3$. We have a free variable so they are not lin indep. But if we use only the first 3 vectors, then we have no free var, and we still have a pivot in each row, so $\begin{pmatrix}1\\-1\\3\end{pmatrix}, \begin{pmatrix}2\\0\\2\end{pmatrix}, \begin{pmatrix}-1\\1\\1\end{pmatrix}$ both span and are lin indep, so they are a basis for $\mathbb{R}^3$.