Solutions

Name:

Math 314/814 Matrix Theory
Final Exam

Show all work. Include all steps necessary to arrive at an answer unaided by a mechanical computational device. The steps you take to your answer are just as important, if not more important, than the answer itself. If you think it, write it!

1. (20 pts.) Find bases for the column space and row space of the matrix

$$A = \begin{pmatrix} 1 & 1 & 2 & 1 \\ 2 & 0 & 6 & 2 \\ 2 & -2 & 12 & -14 \\ -1 & -2 & 1 & -9 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 1 & 2 & 1 \\ 2 & 0 & 6 & 2 \\ 2 & -2 & 12 & -14 \\ -1 & -2 & 1 & -9 \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & 1 & 2 & 1 \\ 0 & -2 & 2 & 0 \\ 0 & -4 & 8 & -16 \\ 0 & -1 & 3 & -8 \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & 1 & 2 & 1 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & +4 & -16 \\ 0 & 0 & 2 & -8 \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & 1 & 2 & 1 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 1 & -4 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\longrightarrow \begin{pmatrix} 1 & 0 & 3 & 1 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 1 & -4 \\ 0 & 0 & 0 & 0 \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & 0 & 0 & 13 \\ 0 & 1 & 0 & -4 \\ 0 & 0 & 1 & -4 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

pivots

So Basis for col(A): $\begin{pmatrix} 1 \\ 2 \\ 2 \\ -1 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ -2 \\ -2 \end{pmatrix}, \begin{pmatrix} 2 \\ 6 \\ 12 \\ -1 \end{pmatrix}$

Basis for row(A): $\begin{pmatrix} 1 \\ 0 \\ 0 \\ 13 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \\ -4 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \\ -4 \end{pmatrix}$

2. (25 pts.) Find a basis for $\mathbb{R}^3$ which includes, among its vectors, a basis for the nullspace of

$$A = \begin{pmatrix} 3 & 1 & 3 \\ 2 & 2 & 2 \end{pmatrix}$$

$$\begin{pmatrix} 3 & 1 & 3 \\ 2 & 2 & 2 \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & 1 & 1 \\ 3 & 1 & 3 \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & 1 & 1 \\ 0 & -2 & 0 \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$$

free

so $x+z=0$ $x=-z$ basis for null(A) : $\begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$
$y=0$

$$\begin{pmatrix} -1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

pivots

So $\begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$ are a basis for $\mathbb{R}^3$

basis for null(A).

3. (20 pts.) Find the inverse of the matrix $A = \begin{pmatrix} 1 & 3 & 2 \\ 3 & 2 & 1 \\ 2 & 3 & 1 \end{pmatrix}$.

$$\left(\begin{array}{ccc|ccc} 1 & 3 & 2 & 1 & 0 & 0 \\ 3 & 2 & 1 & 0 & 1 & 0 \\ 2 & 3 & 1 & 0 & 0 & 1 \end{array}\right) \to \left(\begin{array}{ccc|ccc} 1 & 3 & 2 & 1 & 0 & 0 \\ 0 & -7 & -5 & -3 & 1 & 0 \\ 0 & -3 & -3 & -2 & 0 & 1 \end{array}\right)$$

$$\to \left(\begin{array}{ccc|ccc} 1 & 3 & 2 & 1 & 0 & 0 \\ 0 & 1 & 1 & +2/3 & 0 & -1/3 \\ 0 & -7 & -5 & -3 & 1 & 0 \end{array}\right) \to \left(\begin{array}{ccc|ccc} 1 & 3 & 2 & 1 & 0 & 0 \\ 0 & 1 & 1 & +2/3 & 0 & -1/3 \\ 0 & 0 & 2 & 5/3 & 1 & -7/3 \end{array}\right)$$

$$\to \left(\begin{array}{ccc|ccc} 1 & 3 & 2 & 1 & 0 & 0 \\ 0 & 1 & 0 & -1/6 & -1/2 & 5/6 \\ 0 & 0 & 1 & 5/6 & 1/2 & -7/6 \end{array}\right) \to \left(\begin{array}{ccc|ccc} 1 & 3 & 0 & -2/3 & -1 & +7/3 \\ 0 & 1 & 0 & -1/6 & -1/2 & 5/6 \\ 0 & 0 & 1 & 5/6 & 1/2 & -7/6 \end{array}\right)$$

$$\to \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & -1/6 & 1/2 & -1/6 \\ 0 & 1 & 0 & -1/6 & -1/2 & 5/6 \\ 0 & 0 & 1 & 5/6 & 1/2 & -7/6 \end{array}\right)$$

So $A^{-1} = \begin{pmatrix} -1/6 & 1/2 & -1/6 \\ -1/6 & -1/2 & 5/6 \\ 5/6 & 1/2 & -7/6 \end{pmatrix}$

4. (25 pts.) For the matrix $A = \begin{bmatrix} 9 & -4 \\ 20 & -9 \end{bmatrix}$, what is A^{2008} ?

(Hint: knowing its eigenvalues might help...)

$$\det(A-\lambda I) = \begin{vmatrix} 9-\lambda & -4 \\ 20 & -9-\lambda \end{vmatrix} = (9-\lambda)(-9-\lambda) - (-4)(20)$$

$$= \lambda^2 - 9\lambda + 9\lambda - 81 + 80 = \lambda^2 - 1$$

$$= (\lambda-1)(\lambda+1)$$

So e-values are $\lambda = 1$, $\lambda = -1$ A is diagonalizable!

[Stop! So $AP = PD = P\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ for some invertible P

$A = P\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}P^{-1}$ so $A^{2008} = P\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}^{2008}P^{-1} = P\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}P^{-1}$

$= PP^{-1} = I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$.]

$\lambda = 1$: $\begin{bmatrix} 8 & -4 \\ 20 & -10 \end{bmatrix} \to \begin{bmatrix} 2 & -1 \\ 20 & -10 \end{bmatrix} \to \begin{bmatrix} 1 & -1/2 \\ 0 & 0 \end{bmatrix}$

$\cancel{x = 2y}$ $2x = y$ $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$ e-basis.

$\lambda = -1$: $\begin{bmatrix} 10 & -4 \\ 20 & -8 \end{bmatrix} \to \begin{bmatrix} 5 & -2 \\ 0 & 0 \end{bmatrix}$ $5x = 2y$ $\begin{bmatrix} 2 \\ 5 \end{bmatrix}$

So $AP = PD = P\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$ for $P = \begin{bmatrix} 1 & 2 \\ 2 & 5 \end{bmatrix}$. So $\det P = 5 - 4 = 1$

$A = PDP^{-1}$, so $A^{2008} =$ $P^{-1} = \begin{bmatrix} 5 & -2 \\ -2 & 1 \end{bmatrix}$

$\begin{bmatrix} 1 & 2 \\ 2 & 5 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}^{2008}\begin{bmatrix} 5 & -2 \\ -2 & 1 \end{bmatrix}$

$= \dots = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$.

5. (25 pts.) The vectors $\vec{v}_1 = \begin{bmatrix} 1 \\ 0 \\ 1 \\ 1 \end{bmatrix}$, $\vec{v}_2 = \begin{bmatrix} 2 \\ 0 \\ -1 \\ \cancel{0} \end{bmatrix}$ 0, $\vec{v}_3 = \begin{bmatrix} -1 \\ 0 \\ 1 \\ 0 \end{bmatrix}$ are linearly independent (you need not verify this).

Find the vector in $W = \text{span}(\vec{v}_1, \vec{v}_2, \vec{v}_3)\}$ which is closest to the vector $\vec{b} = \begin{bmatrix} 3 \\ 4 \\ 5 \\ 6 \end{bmatrix}$.

$$W = \text{col}(A) \qquad A = \begin{pmatrix} 1 & 2 & -1 \\ 0 & 0 & 0 \\ 1 & -1 & 1 \\ 1 & \cancel{4}0 & 0 \end{pmatrix}$$

$$\text{closest } \vec{v} = A\vec{x} \quad \text{for} \quad A^TA\vec{x} = A^T\vec{b}$$

$$A^TA = \begin{pmatrix} 1 & 0 & 1 & 1 \\ 2 & 0 & -1 & 0 \\ -1 & 0 & 1 & 0 \end{pmatrix}\begin{pmatrix} 1 & 2 & -1 \\ 0 & 0 & 0 \\ 1 & -1 & 1 \\ 1 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 3 & 1 & 0 \\ 1 & 5 & -3 \\ 0 & -3 & 2 \end{pmatrix}$$

$$A^T\vec{b} = \begin{pmatrix} 1 & 0 & 1 & 1 \\ 2 & 0 & -1 & 0 \\ -1 & 0 & 1 & 0 \end{pmatrix}\begin{pmatrix} 3 \\ 4 \\ 5 \\ 6 \end{pmatrix} = \begin{pmatrix} 14 \\ \cancel{2}1 \\ 2 \end{pmatrix}$$

$$\left(\begin{array}{ccc|c} 3 & 1 & 0 & 14 \\ 1 & 5 & -3 & 1 \\ 0 & -3 & 2 & 2 \end{array}\right) \to \left(\begin{array}{ccc|c} 1 & 5 & -3 & 1 \\ 0 & -14 & 9 & 11 \\ 0 & -3 & 2 & 2 \end{array}\right) \to \left(\begin{array}{ccc|c} 1 & 5 & -3 & 1 \\ 0 & 1 & -2/3 & -2/3 \\ 0 & -14 & 9 & 11 \end{array}\right)$$

$$\to \left(\begin{array}{ccc|c} 1 & 5 & -3 & 1 \\ 0 & 1 & -2/3 & -2/3 \\ 0 & 0 & -1/3 & 5/3 \end{array}\right) \to \left(\begin{array}{ccc|c} 1 & 5 & -3 & 1 \\ 0 & 1 & -2/3 & -2/3 \\ 0 & 0 & 1 & -5 \end{array}\right) \to \left(\begin{array}{ccc|c} 1 & 5 & \cancel{-3}0 & -14 \\ 0 & 1 & 0 & -4 \\ 0 & 0 & 1 & -5 \end{array}\right)$$

$$\to \left(\begin{array}{ccc|c} 1 & 0 & 0 & 6 \\ 0 & 1 & 0 & -4 \\ 0 & 0 & 1 & -5 \end{array}\right) \quad \text{so} \quad \vec{x} = \begin{pmatrix} 6 \\ -4 \\ -5 \end{pmatrix} \quad \text{so}$$

$$\text{closest } \vec{v} = \begin{pmatrix} 1 & 2 & -1 \\ 0 & 0 & 0 \\ 1 & -1 & 1 \\ 1 & 0 & 0 \end{pmatrix}\begin{pmatrix} 6 \\ -4 \\ -5 \end{pmatrix} = \begin{pmatrix} 3 \\ 0 \\ 5 \\ 6 \end{pmatrix}$$

6. (20 pts.) Find the line $y = ax + b$ that best approximates the data points

$$\{(-2,3),(0,5),(1,7)\}\ .$$

$$\begin{pmatrix} -2 & 1 \\ 0 & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} \text{ closest to } \begin{pmatrix} 3 \\ 5 \\ 7 \end{pmatrix} = \vec{b}$$

$$A = \begin{pmatrix} -2 & 1 \\ 0 & 1 \\ 1 & 1 \end{pmatrix} \qquad A^TA = \begin{pmatrix} -2 & 0 & 1 \\ 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} -2 & 1 \\ 0 & 1 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} 5 & -1 \\ -1 & 3 \end{pmatrix}$$

$$A^T\vec{b} = \begin{pmatrix} -2 & 0 & 1 \\ 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} 3 \\ 5 \\ 7 \end{pmatrix} = \begin{pmatrix} 1 \\ 15 \end{pmatrix}$$

$$\det(A^TA) = 5 \cdot 3 - (-1)(-1) = 15 - 1 = 14$$

$$(A^TA)^{-1} = \frac{1}{14} \begin{pmatrix} 3 & 1 \\ 1 & 5 \end{pmatrix}$$

$$\begin{pmatrix} a \\ b \end{pmatrix} = (A^TA)^{-1}(A^T\vec{b}) = \frac{1}{14} \begin{pmatrix} 3 & 1 \\ 1 & 5 \end{pmatrix} \begin{pmatrix} 1 \\ 15 \end{pmatrix}$$

$$\cancel{= \frac{1}{14} \begin{pmatrix} 13 \\ 51 \end{pmatrix} = \begin{pmatrix} 13/14 \\ 51/14 \end{pmatrix}}$$

$$= \frac{1}{14} \begin{pmatrix} 18 \\ 76 \end{pmatrix} = \begin{pmatrix} 9/7 \\ 38/7 \end{pmatrix}$$

$y = \frac{9}{7}a + \frac{38}{7}$ is the best-fitting line.

7. (15 pts.) Show that if A, B are a pair of $m \times n$ matrices, then the collection of vectors

$$W = \{\vec{v} \in \mathbb{R}^n : A\vec{v} = B\vec{v}\}$$

is a subspace of $\mathbb{R}^m$.

$A\vec{v} = B\vec{v} \iff$ $\cancel{(A-B)}$ $A\vec{v} - B\vec{v} = 0 = (A-B)\vec{v}$

so $W = \text{null}(A-B)$, which is a subspace!

or: $\vec{0} \in W$: $A\vec{0} = \vec{0} = B\vec{0}$. ✓

if $\vec{v}, \vec{w} \in W$ then $A\vec{v} = B\vec{v}$, $A\vec{w} = B\vec{w}$, so

$A(\vec{v}+\vec{w}) = A\vec{v} + A\vec{w} = B\vec{v} + B\vec{w} = B(\vec{v}+\vec{w})$, so

$\vec{v}+\vec{w} \in W$.

if $\vec{v} \in W$, $c \in \mathbb{R}$, then $A\vec{v} = B\vec{v}$ so

$A(c\vec{v}) = cA\vec{v} = cB\vec{v} = B(c\vec{v})$ so $c\vec{v} \in W$.

so W is closed under addition an scalar multiplication,

so W is a subspace. ///

8.

Math 314/814, Section 5

Quiz number 11 Solution

Show all work. How you get your answer is just as important, if not more important, than the answer itself. If you think it, write it!

Use Gram-Schmidt orthogonalization, starting with the basis

$$\begin{bmatrix}1\\1\\1\end{bmatrix}, \begin{bmatrix}1\\1\\2\end{bmatrix}, \begin{bmatrix}1\\2\\3\end{bmatrix}$$

for $\mathbb{R}^3$, to build an orthogonal basis for $\mathbb{R}^3$.

Starting with the vectors $\vec{w}_1, \vec{w}_2, \vec{w}_3$ above, we construct:

$$\vec{v}_1 = \vec{w}_1 = \begin{bmatrix}1\\1\\1\end{bmatrix}$$

$$\vec{v}_2 = \vec{w}_2 - \left(\frac{< \vec{w}_2, \vec{v}_1 >}{< \vec{v}_1, \vec{v}_1 >}\vec{v}_1\right) = \begin{bmatrix}1\\1\\2\end{bmatrix} - \frac{4}{3}\begin{bmatrix}1\\1\\1\end{bmatrix} = \begin{bmatrix}-1/3\\-1/3\\2/3\end{bmatrix} = \frac{1}{3}\begin{bmatrix}-1\\-1\\2\end{bmatrix}$$

$$\vec{v}_3 = \vec{w}_3 - \left(\frac{< \vec{w}_3, \vec{v}_1 >}{< \vec{v}_1, \vec{v}_1 >}\vec{v}_1 + \frac{< \vec{w}_3, \vec{v}_2 >}{< \vec{v}_2, \vec{v}_2 >}\vec{v}_2\right) = \begin{bmatrix}1\\2\\3\end{bmatrix} - \left(\frac{6}{3}\begin{bmatrix}1\\1\\1\end{bmatrix} + \frac{\frac{1}{3}3}{(\frac{1}{3})^2(6)}(\frac{1}{3})\begin{bmatrix}-1\\-1\\2\end{bmatrix}\right)$$

$$= \begin{bmatrix}1\\2\\3\end{bmatrix} - \left(2\begin{bmatrix}1\\1\\1\end{bmatrix} + \frac{1}{2}\begin{bmatrix}-1\\-1\\2\end{bmatrix}\right) = \begin{bmatrix}1-2+1/2\\2-2+1/2\\3-2-1\end{bmatrix} = \begin{bmatrix}-1/2\\1/2\\0\end{bmatrix} = \frac{1}{2}\begin{bmatrix}-1\\1\\0\end{bmatrix}$$

So our orthogonal basis is $\begin{bmatrix}1\\1\\1\end{bmatrix}, \frac{1}{3}\begin{bmatrix}-1\\-1\\2\end{bmatrix}, \frac{1}{2}\begin{bmatrix}-1\\1\\0\end{bmatrix}$.

Math 314/814, Section 5

Quiz number 8 Solutions

Show all work. How you get your answer is just as important, if not more important, than the answer itself. If you think it, write it!

9. For what values of x is the matrix

$$A = \begin{pmatrix} x & 1 & 3 \\ 3 & 1 & x \\ 0 & -1 & x \end{pmatrix}$$

invertible?

A is invertible precisely when its determinant is non-zero. So we compute

$\det(A) = x{\cdot}\det\begin{pmatrix} 1 & x \\ -1 & x \end{pmatrix} - 3{\cdot}\det\begin{pmatrix} 1 & 3 \\ -1 & x \end{pmatrix} + 0{\cdot}\det\begin{pmatrix} 1 & 3 \\ 1 & x \end{pmatrix}$
$= x(x-(-x)) - 3(x-(-3)) + 0(x-3) = 2x^2 - 3x - 9$.

To find out where it is non-zero, we find out where it is zero:

$2x^2 - 3x - 9 = 0$ for $x = \dfrac{3 \pm \sqrt{9 + 4\cdot 2\cdot 9}}{2\cdot 2} = \dfrac{3 \pm \sqrt{81}}{4}$,
so $x = (3+9)/4 = 3$ or $x = (3-9)/4 = -3/2$

So, for $x \neq -3/2, 3$, A is invertible.

Math 314/814, Section 5

Quiz number 7 solutions

Show all work. How you get your answer is just as important, if not more important, than the answer itself. If you think it, write it!

10. Find the eigenvalues of the matrix

$$A = \begin{pmatrix} 2 & 3 \\ 4 & 3 \end{pmatrix}$$

and, for each eigenvalue, find a basis for its eigenspace.

To find the eigenvalues, we solve the equation

$$(2-\lambda)(3-\lambda) - (3)(4) = 0 = \lambda^2 - 5\lambda + 6 - 12 = \lambda^2 - 5\lambda - 6 = (\lambda - 6)(\lambda + 1),$$

so $\lambda = 6$ or $\lambda = -1$. These are our eigenvalues.

To find bases for the eigenspaces we find bases for the nullspaces of $A - 6I$ and $A - (-1)I = A + I$, by row reducing:

$$A - 6I = \begin{pmatrix} 2-6 & 3 \\ 4 & 3-6 \end{pmatrix} = \begin{pmatrix} -4 & 3 \\ 4 & -3 \end{pmatrix} \to \begin{pmatrix} 1 & -3/4 \\ 4 & -3 \end{pmatrix} \to \begin{pmatrix} 1 & -3/4 \\ 0 & 0 \end{pmatrix}$$

so the second variable is free and we have $x - (3/4)y = 0$ so $x = (3/4)y$.

So E_6 has basis $\begin{bmatrix} 3/4 \\ 1 \end{bmatrix}$ (or any non-zero multiple of this).

$$A + I = \begin{pmatrix} 2+1 & 3 \\ 4 & +1 \end{pmatrix} = \begin{pmatrix} 3 & 3 \\ 4 & 4 \end{pmatrix} \to \begin{pmatrix} 1 & 1 \\ 4 & 4 \end{pmatrix} \to \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}$$

so the second variable is again free and we have $x + y = 0$ so $x = -y$.

So E_{-1} has basis $\begin{bmatrix} -1 \\ 1 \end{bmatrix}$ (or any non-zero multiple of this).

We can check these answers:

$$A \begin{bmatrix} 3/4 \\ 1 \end{bmatrix} = \begin{bmatrix} 2(3/4) + 3(1) \\ 4(3/4) + 3(1) \end{bmatrix} = \begin{bmatrix} 9/2 \\ 6 \end{bmatrix} = 6 \begin{bmatrix} 3/4 \\ 1 \end{bmatrix}, \text{ and}$$

$$A \begin{bmatrix} -1 \\ 1 \end{bmatrix} = \begin{bmatrix} 2(-1) + 3(1) \\ 4(-1) + 3(1) \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \end{bmatrix} = (-1) \begin{bmatrix} -1 \\ 1 \end{bmatrix}, \text{ as desired.}$$

Math 314/814, Section 5

Quiz number 6 Solutions

Show all work. How you get your answer is just as important, if not more important, than the answer itself. If you think it, write it!

11. 1. Explain why the function $f:\mathbf{R}^2 \to \mathbf{R}^2$ given by
$$f(x,y) = (x-y, x^2+y^2)$$
is **not** a linear transformation.

$f((x,y)+(a,b)) = f(x+a, y+b) = ((x+a)-(y+b), (x+a)^2+(y+b)^2)$
$= ((x-y)+(a-b), (x^2+y^2)+(a^2+b^2)+(2ax+2yb))$
$= (x-y, x^2+y^2)+(a-b, a^2+b^2)+(0, 2ax+2yb) = f(x,y)+f(a,b)+(0,2ax+2yb)$,
which does **not** equal $f(x,y)+f(a,b)$ in general, e.g. for $x=y=a=b=1$.

Alternatively, $f(c(x,y)) = f(cx,cy) = (cx-cy, (cx)^2+(cy)^2) = (cx-cy, c^2x^2+c^2y^2)$, while
$cf(x,y) = c(x-y, x^2+y^2) = (cx-cy, cx^2+cy^2)$, which does **not** equal $f(c(x,y))$ in general, since
$c^2x^2+c^2y^2 \neq cx^2+cy^2$ in general, e.g. for $x=y=1$ and $c=672$.

12. 2. Find the matrix A so that $T = T_A$, where $T:\mathbf{R}^2 \to \mathbf{R}^2$ is the linear transformation which given a vector $[x \quad y]^T$ returns the vector $[y \quad x]^T$. Geometrically, what does this transformation do?

$$T\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} y \\ x \end{bmatrix} = \begin{bmatrix} 0\cdot x + 1\cdot y \\ 1\cdot x + 0\cdot y \end{bmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{bmatrix} x \\ y \end{bmatrix}$$

So $A = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$.

Geometrically, the transformation sends $(0,1)$ to $(1,0)$ and sends $(0,1)$ to $(1,0)$; in general, the transformation swaps the x- and y-coordinates, which can be accomplished by reflecting the vector across the line $y=x$. So T is reflection across the line $y=x$.

Math 314/814, Section 5

Quiz number 5 Solutions

Show all work. How you get your answer is just as important, if not more important, than the answer itself. If you think it, write it!

13. Find bases for the column space of the matrix $A = \begin{pmatrix} 2 & 3 & 4 \\ 5 & 7 & 9 \\ 1 & 4 & 7 \end{pmatrix}$, by

(a) row reducing the matrix A ,

(b) row reducing the transpose A^T of the matrix A .

$$A = \begin{pmatrix} 2 & 3 & 4 \\ 5 & 7 & 9 \\ 1 & 4 & 7 \end{pmatrix} \to \begin{pmatrix} 1 & 4 & 7 \\ 2 & 3 & 4 \\ 1 & 4 & 7 \end{pmatrix} \to \begin{pmatrix} 1 & 4 & 7 \\ 0 & -5 & -10 \\ 0 & -13 & -26 \end{pmatrix}$$

$$\to \begin{pmatrix} 1 & 4 & 7 \\ 0 & 1 & 2 \\ 0 & -13 & -26 \end{pmatrix} \to \begin{pmatrix} 1 & 4 & 7 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{pmatrix}$$

The REF has pivots in the first two columns, so the first two columns of A form a basis for col(A).

$$\text{basis} = \begin{bmatrix} 2 \\ 5 \\ 1 \end{bmatrix}, \begin{bmatrix} 3 \\ 7 \\ 4 \end{bmatrix}$$

$$A^T = \begin{pmatrix} 2 & 5 & 1 \\ 3 & 7 & 4 \\ 4 & 8 & 7 \end{pmatrix} \to \begin{pmatrix} 1 & 5/2 & 1/2 \\ 3 & 7 & 4 \\ 4 & 8 & 7 \end{pmatrix} \to \begin{pmatrix} 1 & 5/2 & 1/2 \\ 0 & -1/2 & 5/2 \\ 0 & -1 & 5 \end{pmatrix}$$

$$\to \begin{pmatrix} 1 & 5/2 & 1/2 \\ 0 & 1 & -5 \\ 0 & -1 & 5 \end{pmatrix} \to \begin{pmatrix} 1 & 0 & 13 \\ 0 & 1 & -5 \\ 0 & 0 & 0 \end{pmatrix}$$

The non-zero rows, $[1 \quad 0 \quad 13]$ and $[0 \quad 1 \quad -5]$, form a basis for row(A^T). So their transposes,

$$\begin{bmatrix} 1 \\ 0 \\ 13 \end{bmatrix} \text{ and } \begin{bmatrix} 0 \\ 1 \\ -5 \end{bmatrix}$$, form a basis for col(A) .

3. (15 pts.) Find the value of Ax closest to b, where

$$A = \begin{pmatrix} 1 & 2 \\ 1 & \cancel{0}\,1 \\ 0 & 1 \end{pmatrix} \qquad \text{and} \qquad b = \begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix}$$

$$A^T A \bar{x} = A^T b$$

$$A^T A = \begin{pmatrix} 1 & 1 & 0 \\ 2 & 1 & 1 \end{pmatrix}\begin{pmatrix} 1 & 2 \\ 1 & 1 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 2 & 3 \\ 3 & 6 \end{pmatrix} \qquad A^T b = \begin{pmatrix} 1 & 1 & 0 \\ 2 & 1 & 1 \end{pmatrix}\begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix} = \begin{pmatrix} 3 \\ 6 \end{pmatrix}$$

$$\left(\begin{array}{cc|c} 2 & 3 & 3 \\ 3 & 6 & 6 \end{array}\right) \to \left(\begin{array}{cc|c} 1 & 2 & 2 \\ 2 & 3 & 3 \end{array}\right)$$

$$\to \left(\begin{array}{cc|c} 1 & 2 & 2 \\ 0 & -1 & -1 \end{array}\right) \to \left(\begin{array}{cc|c} 1 & 0 & 0 \\ 0 & 1 & 1 \end{array}\right) \qquad \begin{matrix} x_1 = 0 \\ x_2 = 1 \end{matrix}$$

$$A\bar{x} = \begin{pmatrix} 1 & 2 \\ 1 & 1 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix}$$

$$(A^T A)^{-1} = \frac{1}{3}\begin{pmatrix} 6 & -3 \\ -3 & 2 \end{pmatrix} \qquad A(A^T A)^{-1} A^T$$

$$= \cancel{\frac{1}{3}\begin{pmatrix} 1 & 1 & 0 \\ 2 & 1 & 1 \end{pmatrix}\begin{pmatrix} 6 & -3 \\ -3 & \end{pmatrix}\cdots}$$

$$= \frac{1}{3}\begin{pmatrix} 1 & 2 \\ 1 & 1 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 6 & -3 \\ -3 & 2 \end{pmatrix}\begin{pmatrix} 1 & 1 & 0 \\ 2 & 1 & 1 \end{pmatrix} = \frac{1}{3}\begin{pmatrix} 1 & 2 \\ 1 & 1 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 0 & 3 & -3 \\ 1 & -1 & 2 \end{pmatrix}$$

$$= \frac{1}{3}\begin{pmatrix} 2 & 1 & 1 \\ 1 & 2 & -1 \\ 1 & -1 & 2 \end{pmatrix} \qquad (\ )b = \frac{1}{3}\begin{pmatrix} 6 \\ 3 \\ 3 \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix}$$

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4. (~~15~~ pts.) Find the determinant of the matrix

$$A = \begin{pmatrix} 1 & 2 & -7 \\ 1 & 0 & -3 \\ 3 & 1 & 3 \end{pmatrix}$$

.

Based on this, find the determinants of the matrices $B = A^{-1}$, $C = A^T$, and $D = A^T A$. (Hint: you don't need to compute these matrices....)

$\det A = 1(0+3) - 1(6+7) + 3(-6-0)$

$= 3 - 13 - 18 = 3 - 31 = -28$

$\det(B) = \frac{1}{\det A} = \frac{1}{-28}$

$\det(C) = \det A) = -28$

$\det(D) = \det(A^T)\det(A) = (-28)^2$

20 a basis for

7. (~~15~~ pts.) Find ~~the~~ a basis for orthogonal complement of the subspace W of R^4 spanned by the vectors

$$\begin{pmatrix} 1 \\ -1 \\ 0 \\ 2 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} -1 \\ 1 \\ 1 \\ 2 \end{pmatrix}$$

$$A = \begin{pmatrix} 1 & -1 \\ -1 & 1 \\ 0 & 1 \\ 2 & 2 \end{pmatrix} \qquad W = C(A)$$

$$W^{\perp} = N(A^T) \qquad A^T = \begin{pmatrix} 1 & -1 & 0 & 2 \\ -1 & 1 & 1 & 2 \end{pmatrix} \to$$

$$\begin{pmatrix} 1 & -1 & 0 & 2 \\ 0 & 0 & 1 & 4 \end{pmatrix}$$

free

$$x_1 = x_2 - 2x_4$$
$$x_3 = -4x_4$$

$$\begin{matrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{matrix} = \begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \end{pmatrix} x_2 + \begin{pmatrix} -2 \\ 0 \\ -4 \\ 1 \end{pmatrix} x_4$$

basis for $W^{\perp}$.