Math 314 Section 4

Quiz number 4 solution

Show all work. How you get your answer is just as important, if not more important, than the answer itself. If you think it, write it!

We can express $\begin{pmatrix}1\\0\end{pmatrix}$ and $\begin{pmatrix}0\\1\end{pmatrix}$ as linear combinations of $\begin{pmatrix}3\\2\end{pmatrix}$ and $\begin{pmatrix}2\\1\end{pmatrix}$, as

$$\begin{pmatrix}1\\0\end{pmatrix} = (-1)\begin{pmatrix}3\\2\end{pmatrix} + 2\begin{pmatrix}2\\1\end{pmatrix} \quad , \quad \begin{pmatrix}0\\1\end{pmatrix} = 2\begin{pmatrix}3\\2\end{pmatrix} + (-3)\begin{pmatrix}2\\1\end{pmatrix}$$

If we are given a **linear transformation** $T : \mathbb{R}^2 \to \mathbb{R}^2$, for which

$$T\begin{pmatrix}3\\2\end{pmatrix} = \begin{pmatrix}1\\1\end{pmatrix} \quad \text{and} \quad T\begin{pmatrix}2\\1\end{pmatrix} = \begin{pmatrix}-1\\1\end{pmatrix},$$

find the matrix A for T, so that $T(\vec{v}) = A\vec{x}$ for every $\vec{x} \in \mathbb{R}^2$.

[Hint: Find $T(\vec{e}_1)$ and $T(\vec{e}_2)$!]

Because T is linear, we know that

$$T\begin{pmatrix}1\\0\end{pmatrix} = T\Big((-1)\begin{pmatrix}3\\2\end{pmatrix} + 2\begin{pmatrix}2\\1\end{pmatrix}\Big) = (-1)T\begin{pmatrix}3\\2\end{pmatrix} + 2T\begin{pmatrix}2\\1\end{pmatrix}$$

$$= (-1)\begin{pmatrix}1\\1\end{pmatrix} + 2\begin{pmatrix}-1\\1\end{pmatrix} = \begin{pmatrix}-3\\1\end{pmatrix}$$

and

$$T\begin{pmatrix}0\\1\end{pmatrix} = T\Big(2\begin{pmatrix}3\\2\end{pmatrix} + (-3)\begin{pmatrix}2\\1\end{pmatrix}\Big) = 2T\begin{pmatrix}3\\2\end{pmatrix} + (-3)T\begin{pmatrix}2\\1\end{pmatrix}$$

$$= 2\begin{pmatrix}1\\1\end{pmatrix} + (-3)\begin{pmatrix}-1\\1\end{pmatrix} = \begin{pmatrix}5\\-1\end{pmatrix}.$$

These vectors form the columns of the matrix A for the transformation T, so

$$A = \begin{pmatrix}-3 & 5\\1 & -1\end{pmatrix}.$$

Note: we can check that this is correct by computing:

$$\begin{pmatrix}-3 & 5\\1 & -1\end{pmatrix}\begin{pmatrix}3\\2\end{pmatrix} = \begin{pmatrix}1\\1\end{pmatrix}, \quad \text{and} \quad \begin{pmatrix}-3 & 5\\1 & -1\end{pmatrix}\begin{pmatrix}2\\1\end{pmatrix} = \begin{pmatrix}-1\\1\end{pmatrix}.$$