Math 314

Exam 2 practice problems

Solutions

Solutions

**Name:**

## Math 314 Matrix Theory

### Exam 2

Show all work. How you get your answer is just as important, if not more important, than the answer itself. If you think it, write it!

**1.** (20 pts.) Find, using any method (other than psychic powers), the determinant of the matrix

$$A = \begin{pmatrix} 1 & 2 & 1 \\ 0 & -1 & 3 \\ 4 & 4 & 2 \end{pmatrix}$$

Is this matrix invertible?

$$\longrightarrow \begin{pmatrix} 1 & 2 & 1 \\ 0 & -1 & 3 \\ 0 & -4 & -2 \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & 2 & 1 \\ 0 & -1 & 3 \\ 0 & 0 & -14 \end{pmatrix}$$

$$\det(A) = 1 \cdot (-1)(-14) = 14$$

$\det(A) \neq 0$ so $A$ is invertible

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or

$$\det(A) = 1 \begin{vmatrix} -1 & 3 \\ 4 & 2 \end{vmatrix} - 0 \begin{vmatrix} 2 & 1 \\ 4 & 2 \end{vmatrix} + 4 \begin{vmatrix} 2 & 1 \\ -1 & 3 \end{vmatrix}$$

$$= 1(-14) + 4(7) = 14.$$

**2.** (15 pts.) Explain why the set of vectors

$$W = \{(x, y, z) \mid x + y + 2z = 1\}$$

is **not** a subspace of $\mathbf{R}^3$.

How many reasons do you want?

$0 + 0 + 2(0) = 0 \neq 1$ so $(0,0,0) \notin W$
so it can't be a subspace.

$\overset{u}{\overset{\shortparallel}{(1,0,0)}}, \overset{v}{\overset{\shortparallel}{(0,1,0)}} \in W$ but:

(a) $u+v = (1,1,0)$ has $1+1+2(0) = 2 \neq 1$ so
$u+v \notin W$, so it can't be...

(b) $2 \cdot u = (2,0,0)$ has $2+0+2(0) = 2 \neq 1$
so $2u \notin W$ so it can't be...

$-u = (-1,0,0)$ has $(-1)+0+2(0) = -1 \neq 1$ so it
can't be....

Any one answer will do!

**4.**(20 pts.) For the matrix

$$A = \begin{pmatrix} 1 & 2 & 1 & 0 \\ 2 & 1 & 3 & 2 \\ 1 & 1 & 1 & 1 \end{pmatrix}$$

find bases for, and the dimensions of, the row, column, and null spaces of $A$.

$$A \longrightarrow \begin{pmatrix} 1 & 2 & 1 & 0 \\ 0 & -3 & 1 & 2 \\ 0 & -1 & 0 & 1 \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & 2 & 1 & 0 \\ 0 & 1 & 0 & -1 \\ 0 & -3 & 1 & 2 \end{pmatrix}$$

$$\longrightarrow \begin{pmatrix} 1 & 2 & 1 & 0 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & -1 \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & 0 & 1 & 2 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & -1 \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & 0 & 0 & 3 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & -1 \end{pmatrix}$$

$\mathcal{R}(A)$ : basis $= \begin{pmatrix} 1 \\ 0 \\ 0 \\ 3 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \\ -1 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \\ -1 \end{pmatrix}$ dim $= \underline{3}$

$\mathcal{C}(A)$ : basis $= \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}, \begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 3 \\ 1 \end{pmatrix}$ dim $= \underline{3}$

$\mathcal{N}(A)$ :

$x_1 + 3x_4 = 0$
$x_2 - x_4 = 0$
$x_3 - x_4 = 0$

$x_1 = -3x_4$
$x_2 = x_4$
$x_3 = x_4$

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = x_4 \begin{pmatrix} -3 \\ 1 \\ 1 \\ 1 \end{pmatrix}$$ dim $= \underline{1}$

↑
basis

**5.** (20 pts.) Find **all** of the solutions to the equation $Ax = b$, where

$$A = \begin{pmatrix} 1 & 2 & 2 & 1 \\ 2 & 4 & 3 & 3 \\ 1 & 2 & 1 & 2 \end{pmatrix} \text{ and } b = \begin{pmatrix} -2 \\ -2 \\ 0 \end{pmatrix}$$

$$\left(\begin{array}{cccc|c} 1 & 2 & 2 & 1 & -2 \\ 2 & 4 & 3 & 3 & -2 \\ 1 & 2 & 1 & 2 & 0 \end{array}\right) \longrightarrow \left(\begin{array}{cccc|c} 1 & 2 & 2 & 1 & -2 \\ 0 & 0 & -1 & 1 & 2 \\ 0 & 0 & -1 & +1 & 2 \end{array}\right)$$

$$\longrightarrow \left(\begin{array}{cccc|c} 1 & 2 & 2 & 1 & -2 \\ 0 & 0 & 1 & -1 & -2 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right) \longrightarrow \left(\begin{array}{cccc|c} 1 & 2 & 0 & 3 & 2 \\ 0 & 0 & 1 & -1 & -2 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right)$$

$$x_1 + 2x_2 + 3x_4 = 2 \qquad x_1 = 2 - 2x_2 - 3x_4$$

$$x_3 - x_4 = -2 \qquad x_3 = -2 + x_4$$

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 2 \\ 0 \\ -2 \\ 0 \end{pmatrix} + x_2 \begin{pmatrix} -2 \\ 1 \\ 0 \\ 0 \end{pmatrix} + x_4 \begin{pmatrix} -3 \\ 0 \\ 1 \\ 1 \end{pmatrix}$$

**5.** A friend of yours runs up to you and says 'Look I've found these three vectors $v_1, v_2, v_3$ in $\mathbf{R}^2$ that are linearly independent!' Explain how you know, without even looking at the vectors, that your friend is wrong (again).

3 vectors in $\mathbb{R}^2$ can't be linearly independent,
because if we write them as columns of a matrix
and row reduce $(v_1\, v_2\, v_3) = A \longrightarrow R$
$R$ ~~can~~ has pivots in different rows, so has at most
2 pivots. Since $R$ has 3 columns, it therefore
has a free variable, so $A\vec{x} = \vec{0}$ has a non-$\vec{0}$
solution. This gives a non-trivial linear combination
$av_1 + bv_2 + cv_3 = \vec{0}$, so the vectors are
linearly dependent!

Solutions

**Name:**

**M314 Matrix Theory**
**Exam 2**

Exams provide you the student with an opportunity to demonstrate your understanding of the techniques presented in the course. So:

**Show all work.** The steps you take to your answer are just as important, if not more important, than the answer itself. If you think it, write it!

**1.** (20 pts.) Find, using any method (other than psychic powers), the determinant of the matrix

$$A = \begin{pmatrix} -1 & 0 & 1 & 2 \\ 3 & -3 & 1 & 1 \\ 0 & 2 & 0 & 3 \\ 1 & -1 & 1 & 1 \end{pmatrix}$$

Is this matrix invertible?

$$|A| = (-1)\begin{vmatrix} 1 & 0 & -1 & -2 \\ 0 & -3 & 4 & 7 \\ 0 & 2 & 0 & 3 \\ 0 & -1 & 2 & 3 \end{vmatrix} \Rightarrow (-1)^3 \begin{vmatrix} 1 & 0 & -1 & -2 \\ 0 & 1 & -2 & -3 \\ 0 & 2 & 0 & 3 \\ 0 & -3 & 4 & 7 \end{vmatrix} = (-1)^3 \begin{vmatrix} 1 & 0 & -1 & -2 \\ 0 & 1 & -2 & -3 \\ 0 & 0 & 4 & 9 \\ 0 & 0 & -2 & -2 \end{vmatrix}$$

$$= (-1)^3(-1)(-2)\begin{vmatrix} 1 & 0 & -1 & -2 \\ 0 & 1 & -2 & -3 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 4 & 9 \end{vmatrix} = (-1)^3(-1)(-2)\begin{vmatrix} 1 & 0 & -1 & -2 \\ 0 & 1 & -2 & -3 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 5 \end{vmatrix}$$

$$= (-2)(1)(1)(1)(5) = -10 \neq 0 \quad \therefore A \text{ is invertible.}$$

or (expand on 3rd row)

$$\det|A| = 0\begin{vmatrix} \cancel{-1} & 0 & 1 & 2 \\ \cancel{3} & -3 & 1 & 1 \\ \cancel{1} & -1 & 1 & 1 \end{vmatrix} - 2\begin{vmatrix} -1 & 1 & 2 \\ 3 & 1 & 1 \\ 1 & 1 & 1 \end{vmatrix} + 0\begin{vmatrix} -1 & 0 & 2 \\ 3 & -3 & 1 \\ 1 & 1 & 1 \end{vmatrix} - 3\begin{vmatrix} -1 & 0 & 1 \\ 3 & -3 & 1 \\ 1 & -1 & 1 \end{vmatrix}$$

$$= -2\left(-1\begin{vmatrix} 1 & 1 \\ 1 & 1 \end{vmatrix} - 1\begin{vmatrix} 3 & 1 \\ 1 & 1 \end{vmatrix} + 2\begin{vmatrix} 3 & 1 \\ 1 & 1 \end{vmatrix}\right) - 3\left(-1\begin{vmatrix} -3 & 1 \\ -1 & 1 \end{vmatrix} - 0\begin{vmatrix} 3 & 1 \\ 1 & 1 \end{vmatrix} + 1\begin{vmatrix} 3 & -3 \\ -1 & 1 \end{vmatrix}\right)$$

$$= -2(0 - 2 + 4) - 3(2 - 0 + 0)$$

$$= -4 - \cancel{30}\,6 = -10$$

or [expand on any other row or column...]

**3.** The system of equations

$$\left(\begin{array}{cccc|cccc} 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 \\ 2 & 2 & 3 & 4 & 0 & 1 & 0 & 0 \\ 3 & 3 & -1 & -6 & 0 & 0 & 1 & 0 \\ 1 & 1 & 2 & 3 & 0 & 0 & 0 & 1 \end{array}\right) \text{ row-reduces to } \left(\begin{array}{cccc|cccc} 1 & 1 & 0 & 0 & 14 & -5 & -1 & 0 \\ 0 & 0 & 1 & 0 & -24 & 9 & 2 & 0 \\ 0 & 0 & 0 & 1 & 11 & -4 & -1 & 0 \\ 0 & 0 & 0 & 0 & 1 & -1 & 0 & 1 \end{array}\right).$$

$A \qquad I \qquad R \qquad Q$

If we call the left-hand side of the first pair of matrices A, use this row-reduction information to find the dimensions and bases for the subspaces Row(A), Nul(A), and Row($A^T$).

(5 pts. for each subspace.)

Row(A) has basis (the transposes of) the non-0 rows of R, so $\begin{pmatrix}1\\1\\0\\0\end{pmatrix}, \begin{pmatrix}0\\0\\1\\0\end{pmatrix}, \begin{pmatrix}0\\0\\0\\1\end{pmatrix}$ are a basis for Row(A)

R has one free variable, so Nul(A) has one basis vector.
y is free.

$x+y=0$
$z=0$
$w=0$

~~gives~~ gives

$x=-y$
$y=y$
$z=0$
$w=0$

$\vec{x} = y\begin{pmatrix}-1\\1\\0\\0\end{pmatrix}$ so

$\begin{pmatrix}-1\\1\\0\\0\end{pmatrix}$ is a basis for Nul(A)

Row($A^T$) = Col(A), and Col(A) has basis the pivot columns of A. Pivots are in columns 1,2, and 4, so $\begin{pmatrix}1\\2\\3\\1\end{pmatrix}, \begin{pmatrix}1\\3\\-1\\2\end{pmatrix}, \begin{pmatrix}1\\4\\-6\\3\end{pmatrix}$ is a basis for Row($A^T$).

**3.** Do the vectors $\begin{pmatrix}1\\-1\\3\end{pmatrix}$, $\begin{pmatrix}2\\0\\2\end{pmatrix}$, $\begin{pmatrix}-1\\1\\1\end{pmatrix}$, and $\begin{pmatrix}1\\3\\5\end{pmatrix}$ span $\mathbf{R}^3$?

Are they linearly independent?
Can you find a subset of this collection of vectors which forms a basis for $\mathrm{R}^3$?
(10 pts. for spanning, 10 pts. for lin indep, 5 pts. for basis.)

Both of the first 2 questions can be answered by row-reducing

$$\begin{pmatrix}1&2&-1&1\\-1&0&1&3\\3&2&1&5\end{pmatrix} \longrightarrow \begin{pmatrix}1&2&-1&1\\0&2&0&4\\0&-4&4&2\end{pmatrix}$$

$$\longrightarrow \begin{pmatrix}1&2&-1&1\\0&1&0&\cancel{4}\,2\\0&-4&4&2\end{pmatrix} \longrightarrow \begin{pmatrix}1&2&-1&1\\0&1&0&2\\0&0&4&10\end{pmatrix} \quad \text{REF}.$$

we have 3 pivots, so we have a pivot in every row, so they span $\mathbb{R}^3$. We have a free variable so they are not lin indep. But if we use only the first 3 vectors, then we have no free var, and we still have a pivot in each row, so $\begin{pmatrix}1\\-1\\3\end{pmatrix}, \begin{pmatrix}2\\0\\2\end{pmatrix}, \begin{pmatrix}-1\\1\\1\end{pmatrix}$ both span and are lin indep, so they are a basis for $\mathbb{R}^3$.

133

Solutions

**Name:**

## Math 314 Matrix Theory
## Exam 2

Show all work. How you get your answer is just as important, if not more important, than the answer itself. If you think it, write it!

**1.** (20 pts.) Find the determinant of the matrix

$$A = \begin{pmatrix} 1 & 2 & 1 & -1 \\ 3 & -1 & 2 & 3 \\ 5 & 4 & 2 & 1 \\ 2 & 4 & 2 & -3 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 2 & 1 & -1 \\ 3 & -1 & 2 & 3 \\ 5 & 4 & 2 & 1 \\ 2 & 4 & 2 & -3 \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & 2 & 1 & -1 \\ 0 & -7 & -1 & 6 \\ 0 & -6 & -3 & 6 \\ 0 & 0 & 0 & -1 \end{pmatrix} \xrightarrow{-1} \begin{pmatrix} 1 & 2 & 1 & -1 \\ 0 & -6 & -3 & 6 \\ 0 & -7 & -1 & 6 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

$$\xrightarrow{-\frac{1}{6}} \begin{pmatrix} 1 & 2 & 1 & -1 \\ 0 & 1 & 1/2 & -1 \\ 0 & -7 & -1 & 6 \\ 0 & 0 & 0 & -1 \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & 2 & 1 & -1 \\ 0 & 1 & 1/2 & -1 \\ 0 & 0 & 5/2 & -1 \\ 0 & 0 & 0 & -1 \end{pmatrix} = R$$

$$\det(R) = (1)(1)\left(\tfrac{5}{2}\right)(-1) = (-1)\left(-\tfrac{1}{6}\right)\det(A), \text{ so}$$

$$\det(A) = (-1)(-6)(1)(1)\left(\tfrac{5}{2}\right)(-1) = 6\left(\tfrac{-5}{2}\right) = \left(\tfrac{-30}{2}\right) = \boxed{-15}$$

**2.** (20 pts.) For the vector space $\mathcal{P}_3$ of polynomials of degree less than or equal to 3, let $T : \mathcal{P}_3 \to \mathbf{R}$ be the function

$$T(p) = p(2) + p(3)\ .$$

Show that $T$ is a *linear transformation*, and find numbers $a, b$, and $c$ so that

$$T(x+a) = T(x^2+b) = T(x^3+c) = 0\ .$$

We want: $T(p+q) = T(p) + T(q)$, $T(cp) = cT(p)$ for $c \in \mathbb{R}$, $p, q \in \mathcal{P}_3$.

But

$$T(p+q) = (p+q)(2) + (p+q)(3)$$
$$= (p(2)+q(2)) + (p(3)+q(3)) = (p(2)+p(3)) + (q(2)+q(3))$$
$$= T(p) + T(q) \ \checkmark$$

$$T(cp) = (cp)(2) + (cp)(3) = c(p(2)) + c(p(3))$$
$$= c(p(2)+p(3)) = cT(p) \ \checkmark$$

So: $T$ is a linear transformation.

$$T(x+a) = (2+a) + (3+a) = 2a+5 = 0 \text{ for } a = \tfrac{-5}{2}$$
$$T(x^2+b) = (4+b) + (9+b) = 2b+13 = 0 \text{ for } b = -13/2$$
$$T(x^3+c) = (8+c) + (27+c) = 2c+35 = 0 \text{ for } c = -35/2$$

So $T(x - \tfrac{5}{2}) = T(x^2 - \tfrac{13}{2}) = T(x^3 - \tfrac{35}{2}) = 0$. ///

**4.** (20 pts.) Show that the collection of vectors $W = \{(a\ b\ c)^T \in \mathbf{R}^3 \ :\ 3a - 2b + c = 0\}$ is a *subspace* of $\mathbf{R}^3$, and find a *basis* for $W$.

Need: $\vec{v}, \vec{w} \in W \Rightarrow \vec{v} + \vec{w} \in W$

$\vec{v} \in W,\ c \in \mathbb{R} \Rightarrow c\vec{v} \in W$

$\vec{v} = \begin{pmatrix} a \\ b \\ c \end{pmatrix}$, $\vec{w} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$, so $3a - 2b + c = 0$, $3x - 2y + z = 0$, then

$\vec{v} + \vec{w} = \begin{pmatrix} a+x \\ b+y \\ c+z \end{pmatrix}$, and $3(a+x) - 2(b+y) + (c+z)$
$= (3a - 2b + c) + (3x - 2y + z) = 0 + 0 = 0$

so $\vec{v} + \vec{w} \in W$. ✓

$k\vec{v} = \begin{pmatrix} ka \\ kb \\ kc \end{pmatrix}$, and $3(ka) - 2(kb) + (kc)$
$= k(3a - 2b + c) = k(0) = 0$

so $k\vec{v} \in W$ ✓ so $W$ is a subspace.

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W looks like a nullspace!

$(3\ {-2}\ 1)\begin{pmatrix} a \\ b \\ c \end{pmatrix} = 0$, so $W = \mathrm{Nul}(3\ {-2}\ 1)$.

Basis: row reduce! $(3\ \ {-2}\ \ 1) \longrightarrow (1\ \ {-\tfrac{2}{3}}\ \ \tfrac{1}{3})$

$x - \tfrac{2}{3}y + \tfrac{1}{3}z = 0$

$x = \tfrac{2}{3}y - \tfrac{1}{3}z$

$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} \tfrac{2}{3}y - \tfrac{1}{3}z \\ y \\ z \end{pmatrix} = y\begin{pmatrix} \tfrac{2}{3} \\ 1 \\ 0 \end{pmatrix} + z\begin{pmatrix} -\tfrac{1}{3} \\ 0 \\ 1 \end{pmatrix}$

Basis: $\begin{pmatrix} \tfrac{2}{3} \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -\tfrac{1}{3} \\ 0 \\ 1 \end{pmatrix}$

148

**5.** (15 pts.) If a $5 \times 8$ matrix $C$ has rank equal to 4, what is the dimension of its nullspace (and why does it have that value?)?

$4 = \text{rank}(C) = \dim(\text{Col}(C)) = \#$ of pivots in (R)REF of $C$. $C$ has 8 columns, so with 4 pivots, this means it has 4 free variables in (R)REF.

But $\dim(\text{Nul}(C)) = \#$ of free variables in (R)REF,

so $\dim(\text{Nul}(C)) = \boxed{4} = 8-4$.



150

140

**3.** (25 pts.) Find bases for the column, row, and nullspaces of the matrix

$$B = \begin{pmatrix} 1 & 2 & 1 & -1 \\ 3 & -1 & 2 & 3 \\ -3 & 8 & -1 & -9 \\ 5 & 3 & 4 & 1 \end{pmatrix}.$$

Row reduce!

$$\begin{pmatrix} 1 & 2 & 1 & -1 \\ 3 & -1 & 2 & 3 \\ -3 & 8 & -1 & -9 \\ 5 & 3 & 4 & 1 \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & 2 & 1 & -1 \\ 0 & -7 & -1 & 6 \\ 0 & 14 & 2 & -12 \\ 0 & -7 & -1 & 6 \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & 2 & 1 & -1 \\ 0 & -7 & -1 & 6 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\longrightarrow \begin{pmatrix} 1 & 2 & 1 & -1 \\ 0 & 1 & 1/7 & -6/7 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & 0 & 5/7 & 5/7 \\ 0 & 1 & 1/7 & -6/7 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

pivots ↑↑    ↑↑ free.

$x + 5/7 z + 5/7 w = 0$

$y + 1/7 z - 6/7 w = 0$

$x = -5/7 z - 5/7 w$

$y = -1/7 z + 6/7 w$

$$\begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix} = \begin{pmatrix} -5/7 z - 5/7 w \\ -1/7 z + 6/7 w \\ z \\ w \end{pmatrix}$$

So: $\begin{pmatrix} 1 \\ 3 \\ -3 \\ 5 \end{pmatrix}, \begin{pmatrix} 2 \\ -1 \\ 8 \\ 3 \end{pmatrix}$ = basis for Col(B)

$\begin{pmatrix} 1 \\ 0 \\ 5/7 \\ 5/7 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 1/7 \\ -6/7 \end{pmatrix}$ = basis for Row(B)

$\begin{pmatrix} -5/7 \\ -1/7 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -5/7 \\ 6/7 \\ 0 \\ 1 \end{pmatrix}$ = basis for Nul(B)