Name: Solutions

Math 314/814, Section 5

Quiz number 1

Show all work. How you get your answer is just as important, if not more important, than the answer itself. If you think it, write it!

Use row reduction to find a solution to the system of equations

$$\begin{aligned} 2x + 4y - z &= 0 \\ x + y + z &= 4 \\ -x - y + 2z &= 2 \end{aligned}$$

Dehydrate:

$$\left(\begin{array}{ccc|c} 2 & 4 & -1 & 0 \\ 1 & 1 & 1 & 4 \\ -1 & -1 & 2 & 2 \end{array}\right) \xrightarrow{E_{12}} \left(\begin{array}{ccc|c} 1 & 1 & 1 & 4 \\ 2 & 4 & -1 & 0 \\ -1 & -1 & 2 & 2 \end{array}\right)$$

$$\xrightarrow[E_{31}(1)]{E_{21}(-2)} \left(\begin{array}{ccc|c} 1 & 1 & 1 & 4 \\ 0 & 2 & -3 & -8 \\ 0 & 0 & 3 & 6 \end{array}\right) \xrightarrow{E_3(\frac{1}{3})} \left(\begin{array}{ccc|c} 1 & 1 & 1 & 4 \\ 0 & 2 & -3 & -8 \\ 0 & 0 & 1 & 2 \end{array}\right)$$

REF!

$$\xrightarrow[E_{13}(-1)]{E_{23}(3)} \left(\begin{array}{ccc|c} 1 & 1 & 0 & 2 \\ 0 & 2 & 0 & -2 \\ 0 & 0 & 1 & 2 \end{array}\right) \xrightarrow{E_2(\frac{1}{2})} \left(\begin{array}{ccc|c} 1 & 1 & 0 & 2 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 2 \end{array}\right) \xrightarrow{E_{12}(-1)} \left(\begin{array}{ccc|c} 1 & 0 & 0 & 3 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 2 \end{array}\right)$$

RREF!

rehydrate!

$$\boxed{\begin{aligned} x &= 3 \\ y &= -1 \\ z &= 2 \end{aligned}}$$

Solution!

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Quiz number 2

Show all work. How you get your answer is just as important, if not more important, than the answer itself. If you think it, write it!

Show that the vector $\begin{bmatrix} 6 \\ 3 \\ 16 \\ 0 \end{bmatrix}$ is in the span of the vectors $\begin{bmatrix} 2 \\ 1 \\ 3 \\ 1 \end{bmatrix}$, $\begin{bmatrix} -1 \\ 3 \\ 2 \\ -1 \end{bmatrix}$, and $\begin{bmatrix} 1 \\ 4 \\ -2 \\ 3 \end{bmatrix}$.

(Express it as a linear combination.)

We want that $x\begin{bmatrix} 2 \\ 1 \\ 3 \\ 1 \end{bmatrix} + y\begin{bmatrix} -1 \\ 3 \\ 2 \\ -1 \end{bmatrix} + z\begin{bmatrix} 1 \\ 4 \\ -2 \\ 3 \end{bmatrix} = \begin{bmatrix} 6 \\ 3 \\ 16 \\ 0 \end{bmatrix}$ has a solution.

So we row reduce:

$$\left(\begin{array}{ccc|c} 2 & -1 & 1 & 6 \\ 1 & 3 & 4 & 3 \\ 3 & 2 & -2 & 16 \\ 1 & -1 & 3 & 0 \end{array}\right) \xrightarrow{E_{14}} \left(\begin{array}{ccc|c} 1 & -1 & 3 & 0 \\ 1 & 3 & 4 & 3 \\ 3 & 2 & -2 & 16 \\ 2 & -1 & 1 & 6 \end{array}\right) \xrightarrow[E_{14}(-2)]{E_{12}(-1),\ E_{13}(-3)} \left(\begin{array}{ccc|c} 1 & -1 & 3 & 0 \\ 0 & 4 & 1 & 3 \\ 0 & 5 & -11 & 16 \\ 0 & 1 & -5 & 6 \end{array}\right)$$

$$\xrightarrow{E_{24}} \left(\begin{array}{ccc|c} 1 & -1 & 3 & 0 \\ 0 & 1 & -5 & 6 \\ 0 & 5 & -11 & 16 \\ 0 & 4 & 1 & 3 \end{array}\right) \xrightarrow[E_{24}(-4)]{E_{23}(-5)} \left(\begin{array}{ccc|c} 1 & -1 & 3 & 0 \\ 0 & 1 & -5 & 6 \\ 0 & 0 & 14 & -14 \\ 0 & 0 & 21 & -21 \end{array}\right) \xrightarrow[E_{34}(-21)]{E_3(\frac{1}{14})} \left(\begin{array}{ccc|c} 1 & -1 & 3 & 0 \\ 0 & 1 & -5 & 6 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

REF

consistent! There is a solution.

$$\xrightarrow[E_{31}(-3)]{E_{32}(5)} \left(\begin{array}{ccc|c} 1 & -1 & 0 & 3 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 \end{array}\right) \xrightarrow{E_{21}(1)} \left(\begin{array}{ccc|c} 1 & 0 & 0 & 4 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 \end{array}\right) \Longrightarrow \begin{array}{l} x=4 \\ y=1 \\ z=-1 \end{array}$$

So $\boxed{\begin{pmatrix} 6 \\ 3 \\ 16 \\ 0 \end{pmatrix} = 4\begin{pmatrix} 2 \\ 1 \\ 3 \\ 1 \end{pmatrix} + 1\begin{pmatrix} -1 \\ 3 \\ 2 \\ -1 \end{pmatrix} + (-1)\begin{pmatrix} 1 \\ 4 \\ -2 \\ 3 \end{pmatrix}}$

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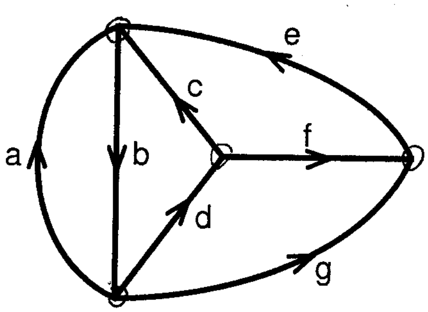
Quiz number 3

Show all work. How you get your answer is just as important, if not more important, than the answer itself. If you think it, write it!

Water is flowing through a network of pipes represented by the graph below. (Labels and directions have been provided for the sanity of your instructor when grading this problem.) Identify a (smallest) collection of pipes to monitor, in order to determine the flow rate through every pipe. If the flow rates through all of your chosen pipes are equal to +1, what are the flow rates through each of the other pipes?



rates in = rates out

$a+c+e=b$ $\qquad$ $a-b+c+e=0$

$b=a+d+g$ $\qquad$ $a-b+d+g=0$

$d=c+f$ $\qquad$ $c-d+f=0$

$f+g=e$ $\qquad$ $e-f-g=0$

$$\begin{array}{c}
\begin{array}{ccccccc} a & b & c & d & e & f & g \end{array} \\
\left(\begin{array}{ccccccc|c} 1 & -1 & 1 & 0 & 1 & 0 & 0 & 0 \\ 1 & -1 & 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & -1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & -1 & -1 & 0 \end{array}\right)
\end{array}
\xrightarrow{R_2 \leftarrow R_2 - R_1}
\left(\begin{array}{ccccccc|c} 1 & -1 & 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 1 & -1 & 0 & 1 & 0 \\ 0 & 0 & 1 & -1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & -1 & -1 & 0 \end{array}\right)$$

$$\underset{R_2 \leftrightarrow -R_2}{\xrightarrow{R_3 \leftarrow R_3 + R_2}}
\left(\begin{array}{ccccccc|c} 1 & -1 & 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & -1 & 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & -1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & -1 & -1 & 0 \end{array}\right)
\underset{R_3 \leftarrow -R_3}{\xrightarrow{R_4 \leftarrow R_4 + R_3}}
\begin{array}{c}
\begin{array}{ccccccc} & b & & d & & f & g \end{array} \\
\left(\begin{array}{ccccccc|c} 1 & -1 & 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & -1 & 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 1 & -1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right)
\end{array}$$

REF!

Consistent! Free variables = b, d, f, g → these values determine a, c, e

So: monitor $\boxed{b, d, f, g}$.

$1=b=d=f=g$? Continue to RREF. or: rehydrate! $e=f+g=\underline{\underline{2}}$

$c=d-e+g=1-2+1=\underline{\underline{0}}$

$a=b-c-e=1-0-2=\underline{\underline{-1}}$

$$\underset{R_1 \leftarrow R_1 - R_3}{\xrightarrow{R_2 \leftarrow R_2 - R_3}}
\left(\begin{array}{ccccccc|c} 1 & -1 & 1 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & -1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & -1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right)$$

$$\xrightarrow{R_1 \leftarrow R_1 - R_2}
\left(\begin{array}{ccccccc|c} 1 & -1 & 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & -1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & -1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right)
\Longrightarrow
\begin{aligned} a &= b-d-g = 1-1-1 = -1 \\ c &= d-f = 1-1 = 0 \\ e &= f+g = 1+1 = 2 \end{aligned}$$

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Quiz number 4

Show all work. How you get your answer is just as important, if not more important, than the answer itself. If you think it, write it!

Use the superaugmented matrix of the matrix

$A = \begin{pmatrix} 2 & 2 & 3 \\ 3 & 2 & 4 \\ 1 & 2 & 1 \end{pmatrix}$, to find solutions to the equation $A\vec{x} = \vec{b}$, for $\vec{b} = \begin{bmatrix} 1 \\ 3 \\ 4 \end{bmatrix}$ and $\begin{bmatrix} 3 \\ 0 \\ 3 \end{bmatrix}$.

Row reduce!

$$\left(\begin{array}{ccc|ccc} 2&2&3&1&0&0 \\ 3&2&4&0&1&0 \\ 1&2&1&0&0&1 \end{array}\right) \xrightarrow{R_{13}} \left(\begin{array}{ccc|ccc} 1&2&1&0&0&1 \\ 3&2&4&0&1&0 \\ 2&2&3&1&0&0 \end{array}\right) \xrightarrow[R_{31}(-2)]{R_{21}(-3)} \left(\begin{array}{ccc|ccc} 1&2&1&0&0&1 \\ 0&-4&1&0&1&-3 \\ 0&-2&1&1&0&-2 \end{array}\right)$$

$$\xrightarrow{R_{23}} \left(\begin{array}{ccc|ccc} 1&2&1&0&0&1 \\ 0&-2&1&1&0&-2 \\ 0&-4&1&0&1&-3 \end{array}\right) \xrightarrow{R_2(-\frac{1}{2})} \left(\begin{array}{ccc|ccc} 1&2&1&0&0&1 \\ 0&1&-\frac{1}{2}&-\frac{1}{2}&0&1 \\ 0&-4&1&0&1&-3 \end{array}\right) \xrightarrow{R_{32}(4)} \left(\begin{array}{ccc|ccc} 1&2&1&0&0&1 \\ 0&1&-\frac{1}{2}&-\frac{1}{2}&0&1 \\ 0&0&-1&-2&1&1 \end{array}\right)$$

$$\xrightarrow{R_3(-1)} \left(\begin{array}{ccc|ccc} 1&2&1&0&0&1 \\ 0&1&-\frac{1}{2}&-\frac{1}{2}&0&1 \\ 0&0&1&2&-1&-1 \end{array}\right) \xrightarrow[R_{13}(-1)]{R_{23}(\frac{1}{2})} \left(\begin{array}{ccc|ccc} 1&2&0&-2&1&2 \\ 0&1&0&\frac{1}{2}&-\frac{1}{2}&\frac{1}{2} \\ 0&0&1&2&-1&-1 \end{array}\right) \xrightarrow{R_{12}(-2)} \left(\begin{array}{ccc|ccc} 1&0&0&-3&2&1 \\ 0&1&0&\frac{1}{2}&-\frac{1}{2}&\frac{1}{2} \\ 0&0&1&2&-1&-1 \end{array}\right)$$

So $A\vec{x}=\vec{b} \rightarrow \begin{pmatrix} 1&0&0 \\ 0&1&0 \\ 0&0&1 \end{pmatrix}\vec{x} = \vec{x} = \begin{pmatrix} -3&2&1 \\ \frac{1}{2}&-\frac{1}{2}&\frac{1}{2} \\ 2&-1&-1 \end{pmatrix}\vec{b}$ So!

$A\vec{x} = \begin{pmatrix} 1 \\ 3 \\ 4 \end{pmatrix}$ has solution $\vec{x} = \begin{pmatrix} -3&2&1 \\ \frac{1}{2}&-\frac{1}{2}&\frac{1}{2} \\ 2&-1&-1 \end{pmatrix}\begin{pmatrix} 1 \\ 3 \\ 4 \end{pmatrix} = \begin{pmatrix} -3+6+4 \\ \frac{1}{2}-\frac{3}{2}+2 \\ 2-3-4 \end{pmatrix} = \begin{pmatrix} 7 \\ 1 \\ -5 \end{pmatrix}$

and

$A\vec{x} = \begin{pmatrix} 3 \\ 0 \\ 3 \end{pmatrix}$ has solution $\vec{x} = \begin{pmatrix} -3&2&1 \\ \frac{1}{2}&-\frac{1}{2}&\frac{1}{2} \\ 2&-1&-1 \end{pmatrix}\begin{pmatrix} 3 \\ 0 \\ 3 \end{pmatrix} = \begin{pmatrix} -9+0+3 \\ \frac{3}{2}+0+\frac{3}{2} \\ 6+0-3 \end{pmatrix} = \begin{pmatrix} -6 \\ 3 \\ 3 \end{pmatrix}$

Check! $\begin{pmatrix} 2&2&3 \\ 3&2&4 \\ 1&2&1 \end{pmatrix}\begin{pmatrix} 7 \\ 1 \\ -5 \end{pmatrix} = \begin{pmatrix} 14+2-15 \\ 21+2-20 \\ 7+2-5 \end{pmatrix} = \begin{pmatrix} 1 \\ 3 \\ 4 \end{pmatrix}$ ✓!, $\begin{pmatrix} 2&2&3 \\ 3&2&4 \\ 1&2&1 \end{pmatrix}\begin{pmatrix} -6 \\ 3 \\ 3 \end{pmatrix} = \begin{pmatrix} -12+6+9 \\ -18+6+12 \\ -6+6+3 \end{pmatrix} = \begin{pmatrix} 3 \\ 0 \\ 3 \end{pmatrix}$ ✓!

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Quiz number 5

Show all work. How you get your answer is just as important, if not more important, than the answer itself. If you think it, write it!

Find bases for the column space of the matrix $A = \begin{pmatrix} 3 & 1 & -5 \\ 1 & 2 & 5 \\ 3 & 4 & 7 \end{pmatrix}$, by

(a) row reducing the matrix A ,

(b) row reducing the transpose A^T of the matrix A .

$$A = \begin{pmatrix} 3 & 1 & -5 \\ 1 & 2 & 5 \\ 3 & 4 & 7 \end{pmatrix} \to \begin{pmatrix} 1 & 2 & 5 \\ 3 & 1 & -5 \\ 3 & 4 & 7 \end{pmatrix} \to \begin{pmatrix} 1 & 2 & 5 \\ 0 & -5 & -20 \\ 0 & -2 & -8 \end{pmatrix} \to \begin{pmatrix} 1 & 2 & 5 \\ 0 & 1 & 4 \\ 0 & -2 & -8 \end{pmatrix}$$

$$\to \underbrace{\begin{pmatrix} 1 & 2 & 5 \\ 0 & 1 & 4 \\ 0 & 0 & 0 \end{pmatrix}}_{\text{REF!}} \to \begin{pmatrix} 1 & 0 & -3 \\ 0 & 1 & 4 \\ 0 & 0 & 0 \end{pmatrix}$$ pivots in col's #1 & #2

so $\begin{pmatrix} 3 \\ 1 \\ 3 \end{pmatrix}, \begin{pmatrix} 1 \\ 2 \\ 4 \end{pmatrix}$ form a basis ~~for~~ for col(A)

$$A^T = \begin{pmatrix} 3 & 1 & 3 \\ 1 & 2 & 4 \\ -5 & 5 & 7 \end{pmatrix} \to \begin{pmatrix} 1 & 2 & 4 \\ 3 & 1 & 3 \\ -5 & 5 & 7 \end{pmatrix} \to \begin{pmatrix} 1 & 2 & 4 \\ 0 & -5 & -9 \\ 0 & 15 & 27 \end{pmatrix} \to \begin{pmatrix} 1 & 2 & 4 \\ 0 & 1 & 9/5 \\ 0 & 15 & 27 \end{pmatrix}$$

$$\to \underbrace{\begin{pmatrix} 1 & 2 & 4 \\ 0 & 1 & 9/5 \\ 0 & 0 & 0 \end{pmatrix}}_{\text{REF!}} \to \underbrace{\begin{pmatrix} 1 & 0 & 2/5 \\ 0 & 1 & 9/5 \\ 0 & 0 & 0 \end{pmatrix}}_{\text{RREF!}}$$

so transposes of non-zero rows of REF or RREF are a basis for row(A^T) = col(A)

so (REF) $\begin{pmatrix} 1 \\ 2 \\ 4 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 9/5 \end{pmatrix}$ are a basis for row(A^T) = col(A)

or (RREF) $\begin{pmatrix} 1 \\ 0 \\ 2/5 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 9/5 \end{pmatrix}$ are a basis for row(A^T) = col(A).