Math 314/814 Matrix Theory

Solutions to Exam 2 practice problems

1. Is the collection of vectors

$$\left\{ \begin{bmatrix} x \\ y \\ z \end{bmatrix} \in \mathbb{R}^3 : xy + z = 0 \right\}$$

a subspace of $\mathbb{R}^3$? Explain why or why not.

W is not a subspace. For example, $\vec{v} = \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix} \in W$, since $(1)(1) + (-1) = 0$, but $2\vec{v} = \begin{bmatrix} 2 \\ 2 \\ -2 \end{bmatrix} \notin W$, since $(2)(2) + (-2) = 4 - 2 = 2 \neq 0$.

Another demonstration: $\vec{v} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ and $\vec{w} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$ are both in W, but $\vec{v} + \vec{w} = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$ is not in W.

2. Find a basis for $\mathbb{R}^3$ that **includes** the vectors $\begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix}$ and $\begin{bmatrix} 1 \\ 2 \\ 5 \end{bmatrix}$.

To extend $\vec{v}_1$ and $\vec{v}_2$ to a basis for all of $\mathbb{R}^3$, we can add any basis to the collection and then use row reduction to pick a basis from among the larger set. If we make sure that $\vec{v}_1$ and $\vec{v}_2$ are at the beginning of the list, row reduction will produce pivots in those columns, so we will choose those vectors. Any basis will do; we might as well choose $\vec{e}_1, \vec{e}_2$, and $\vec{e}_3$! So we row reduce:

$$A = \begin{bmatrix} 2 & 1 & 1 & 0 & 0 \\ 1 & 2 & 0 & 1 & 0 \\ 1 & 5 & 0 & 0 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & 2 & 0 & 1 & 0 \\ 0 & -3 & 1 & -2 & 0 \\ 0 & 3 & 0 & -1 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & 2 & 0 & 1 & 0 \\ 0 & 3 & 0 & -1 & 1 \\ 0 & -3 & 1 & -2 & 0 \end{bmatrix} \to \begin{bmatrix} 1 & 2 & 0 & 1 & 0 \\ 0 & 3 & 0 & -1 & 1 \\ 0 & 0 & 1 & -3 & 1 \end{bmatrix}$$

at which point we can **stop**; there are pivots in each of the first three columns. Since the last three columns span $\mathbb{R}^3$, $\text{col}(A) = \mathbb{R}^3$, and the first three columns, the columns corresponding to the pivots, form a basis for $\text{col}(A) = \mathbb{R}^3$. So a basis for $\mathbb{R}^3$ is given by

$$\begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 5 \end{bmatrix}, \text{ and } \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}.$$

Alternatively, we could manufacture **any** vector $\vec{v}_3$ which does not lie in the span of $\vec{v}_1$ and $\vec{v}_2$; then $\vec{v}_1$, $\vec{v}_2$ and $\vec{v}_3$ are three linearly independent vectors in $\mathbb{R}^3$, and so form a basis for $\mathbb{R}^3$.

3. Find the **dimensions** of the row, column, and nullspaces of the matrix

$$B = \begin{bmatrix} 1 & -1 & 1 & 3 & -1 \\ 1 & 0 & 2 & 1 & 0 \\ 1 & 1 & 3 & 0 & 0 \\ 1 & 2 & 4 & 1 & -2 \end{bmatrix}$$

To compute the dimensions, we really only need to know how many pivots the matrix has. So we row reduce:

$$B = \begin{bmatrix} 1 & -1 & 1 & 3 & -1 \\ 1 & 0 & 2 & 1 & 0 \\ 1 & 1 & 3 & 0 & 0 \\ 1 & 2 & 4 & 1 & -2 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 2 & 1 & 0 \\ 1 & -1 & 1 & 3 & -1 \\ 1 & 1 & 3 & 0 & 0 \\ 1 & 2 & 4 & 1 & -2 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 2 & 1 & 0 \\ 0 & -1 & -1 & 2 & -1 \\ 0 & 1 & 1 & -1 & 0 \\ 0 & 2 & 2 & 0 & -2 \end{bmatrix} \to$$

$$\begin{bmatrix} 1 & 0 & 2 & 1 & 0 \\ 0 & 1 & 1 & -1 & 0 \\ 0 & -1 & -1 & 2 & -1 \\ 0 & 2 & 2 & 0 & -2 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 2 & 1 & 0 \\ 0 & 1 & 1 & -1 & 0 \\ 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 2 & -2 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 2 & 1 & 0 \\ 0 & 1 & 1 & -1 & 0 \\ 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

This is enough; we see that the matrix B has an REF with three pivots. Consequently, since it has five columns, it has two free variables, and

$$\dim(\mathrm{col}(B)) = \dim(\mathrm{row}(B)) = 3 \qquad \text{and} \qquad \dim(\mathrm{null}(B)) = 2.$$

4. Find the determinant of the matrix $B = \begin{bmatrix} 3 & 1 & 0 \\ 1 & 5 & 2 \\ 0 & 1 & 3 \end{bmatrix}$.

Many methods work fine; here let's expand on the first row:

$$\det(B) = (3)\begin{vmatrix} 5 & 2 \\ 1 & 3 \end{vmatrix} - (1)\begin{vmatrix} 1 & 2 \\ 0 & 3 \end{vmatrix} + (0)\begin{vmatrix} 1 & 5 \\ 0 & 1 \end{vmatrix}$$

$$= (3)\{15-2\} - (1)\{3-0\} + (0) = 39 - 3 = 36.$$

5. The matrix $A = \begin{bmatrix} -1 & 1 & 3 & -1 \\ -1 & 1 & 1 & 1 \\ -2 & 1 & 4 & -1 \\ -1 & 0 & 1 & 2 \end{bmatrix}$ has characteristic polynomial

$$\chi_A(\lambda) = (\lambda-1)^2(\lambda-2)^2.$$

Find the eigenvalues of A and bases for each of the corresponding eigenspaces. Is A diagonalizable? Why or why not?

The eigenvalues are the roots of $\chi_A(\lambda)$, which are $\lambda = 1, 2$. Then we row reduce:

$$A - (1)I = \begin{bmatrix} -2 & 1 & 3 & -1 \\ -1 & 0 & 1 & 1 \\ -2 & 1 & 3 & -1 \\ -1 & 0 & 1 & 1 \end{bmatrix} \to \begin{bmatrix} -1 & 0 & 1 & 1 \\ -2 & 1 & 3 & -1 \\ -2 & 1 & 3 & -1 \\ -1 & 0 & 1 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & -1 & -1 \\ -2 & 1 & 3 & -1 \\ -2 & 1 & 3 & -1 \\ -1 & 0 & 1 & 1 \end{bmatrix} \to$$

$$\begin{bmatrix} 1 & 0 & -1 & -1 \\ 0 & 1 & 1 & -3 \\ 0 & 1 & 1 & -3 \\ 0 & 0 & 0 & 0 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & -1 & -1 \\ 0 & 1 & 1 & -3 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

so the last two variables z, w are free, $x = z + w$ and $y = -z + 3w$, so

$$\begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix} = \begin{bmatrix} z+w \\ -z+3w \\ z \\ w \end{bmatrix} = z \begin{bmatrix} 1 \\ -1 \\ 1 \\ 0 \end{bmatrix} + w \begin{bmatrix} 1 \\ 3 \\ 0 \\ 1 \end{bmatrix}, \text{ and so } \begin{bmatrix} 1 \\ -1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 3 \\ 0 \\ 1 \end{bmatrix} \text{ form a basis for the}$$

$\lambda = 1$ eigenspace.

$$A - (2)I = \begin{bmatrix} -3 & 1 & 3 & -1 \\ -1 & -1 & 1 & 1 \\ -2 & 1 & 2 & -1 \\ -1 & 0 & 1 & 0 \end{bmatrix} \to \begin{bmatrix} -1 & 0 & 1 & 0 \\ -3 & 1 & 3 & -1 \\ -1 & -1 & 1 & 1 \\ -2 & 1 & 2 & -1 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & -1 & 0 \\ -3 & 1 & 3 & -1 \\ -1 & -1 & 1 & 1 \\ -2 & 1 & 2 & -1 \end{bmatrix} \to$$

$$\begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & -1 \\ 0 & -1 & 0 & 1 \\ 0 & 1 & 0 & -1 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

so the last two variables z, w are also free, $x = z$ and $y = -w$, so

$$\begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix} = \begin{bmatrix} z \\ w \\ z \\ w \end{bmatrix} = z \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix} + w \begin{bmatrix} 0 \\ 1 \\ 0 \\ 1 \end{bmatrix}, \text{ and so } \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \\ 1 \end{bmatrix} \text{ form a basis for the } \lambda = 2$$

eigenspace.

And since the geometric multiplicity of each eigenvalue equals 2, which equals the algebraic multiplicity of each eigenvalue, the matrix A is diagonalizable!

6. Show that if $\vec{v} \neq \vec{0}$ is an eigenvector for two $n \times n$ matrices A and B, then $\vec{v}$ is **also** an eigenvector for both of the product matrices AB and BA. Show that under these circumstances we can conclude that the matrix $AB - BA$ is **not** invertible.

We are told that $A\vec{v} = \lambda_1\vec{v}$ and $B\vec{v} = \lambda_2\vec{v}$ for some λ_1, λ_2. So

$$AB\vec{v} = A(B\vec{v}) = A(\lambda_2\vec{v}) = \lambda_2(A\vec{v}) = \lambda_2(\lambda_1\vec{v}) = (\lambda_1\lambda_2)\vec{v}, \text{ and}$$

$$BA\vec{v} = B(A\vec{v}) = B(\lambda_1\vec{v}) = \lambda_1(B\vec{v}) = \lambda_1(\lambda_2\vec{v}) = (\lambda_1\lambda_2)\vec{v},$$

so $\vec{v}$ is also an eigenvector for both AB and BA. But then $(AB-BA)\vec{v} = AB\vec{v}-BA\vec{v} = (\lambda_1\lambda_2)\vec{v}-(\lambda_1\lambda_2)\vec{v} = \vec{0}$, so $AB-BA$ has non-trivial nullspace, and so it cannot be invertible.