Solutions 1:18 (11:43)

Name:

Math 314/814 Matrix Theory, Section 001

Final Exam

Show all work. Include all steps necessary to arrive at an answer unaided by a mechanical computational device. The steps you take to your answer are just as important, if not more important, than the answer itself. If you think it, write it!

A.1. (20 pts.) Find the inverse of the matrix $A = \begin{pmatrix} 2 & 3 & 2 \\ 3 & 2 & -1 \\ 1 & 2 & 2 \end{pmatrix}$.

$$\left(\begin{array}{ccc|ccc} 2 & 3 & 2 & 1 & 0 & 0 \\ 3 & 2 & -1 & 0 & 1 & 0 \\ 1 & 2 & 2 & 0 & 0 & 1 \end{array}\right) \to \left(\begin{array}{ccc|ccc} 1 & 2 & 2 & 0 & 0 & 1 \\ 2 & 3 & 2 & 1 & 0 & 0 \\ 3 & 2 & -1 & 0 & 1 & 0 \end{array}\right)$$

$$\to \left(\begin{array}{ccc|ccc} 1 & 2 & 2 & 0 & 0 & 1 \\ 0 & -1 & -2 & 1 & 0 & -2 \\ 0 & -4 & -7 & 0 & 1 & -3 \end{array}\right) \to \left(\begin{array}{ccc|ccc} 1 & 2 & 2 & 0 & 0 & 1 \\ 0 & 1 & 2 & -1 & 0 & 2 \\ 0 & 0 & 1 & -4 & 1 & 5 \end{array}\right)$$

$$\to \left(\begin{array}{ccc|ccc} 1 & 2 & 0 & 8 & -2 & -9 \\ 0 & 1 & 0 & 7 & -2 & -8 \\ 0 & 0 & 1 & -4 & 1 & 5 \end{array}\right) \to \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & -6 & 2 & 7 \\ 0 & 1 & 0 & 7 & -2 & -8 \\ 0 & 0 & 1 & -4 & 1 & 5 \end{array}\right)$$

So $A^{-1} = \begin{pmatrix} -6 & 2 & 7 \\ 7 & -2 & -8 \\ -4 & 1 & 5 \end{pmatrix}$ //

A.2. (15 pts.) Determine whether or not the vectors

$$\begin{bmatrix}1\\1\\5\\0\end{bmatrix}, \begin{bmatrix}1\\3\\-3\\1\end{bmatrix}, \begin{bmatrix}-1\\-1\\-1\\3\end{bmatrix}, \begin{bmatrix}0\\1\\2\\5\end{bmatrix}$$

are linearly independent.

$$\begin{bmatrix}1 & 1 & -1 & 0\\1 & 3 & -1 & 1\\5 & -3 & -1 & 2\\0 & 1 & 3 & 5\end{bmatrix} \to \begin{bmatrix}1 & 1 & -1 & 0\\0 & 2 & 0 & 1\\0 & -8 & 4 & 2\\0 & 1 & 3 & 5\end{bmatrix}$$

$$\to \begin{bmatrix}1 & 1 & -1 & 0\\0 & 1 & 3 & 5\\0 & 2 & 0 & 1\\0 & -8 & 4 & 2\end{bmatrix} \to \begin{bmatrix}1 & 1 & -1 & 0\\0 & 1 & 3 & 5\\0 & 0 & -6 & -9\\0 & 0 & 28 & 42\end{bmatrix}$$

$$\to \begin{bmatrix}1 & 1 & -1 & 0\\0 & 1 & 3 & 5\\0 & 0 & 1 & 3/2\\0 & 0 & 0 & 0\end{bmatrix}$$

free!

So not lin indep

A.3. (20 pts.) Find an orthogonal basis for the column space of the matrix

$$A = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$$

$$w_1 = \begin{pmatrix} 1 \\ 1 \\ 0 \\ 1 \end{pmatrix} = v_1$$

$$w_2 = \begin{pmatrix} 1 \\ 0 \\ 1 \\ 1 \end{pmatrix} - \frac{1+0+0+1}{1+1+0+1}\begin{pmatrix} 1 \\ 1 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 1 \\ 1 \end{pmatrix} - \frac{2}{3}\begin{pmatrix} 1 \\ 1 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 1/3 \\ -2/3 \\ 1 \\ 1/3 \end{pmatrix}$$

$$w_3 = \begin{pmatrix} 0 \\ 1 \\ 1 \\ 1 \end{pmatrix} - \frac{0+1+0+1}{1+1+0+1}\begin{pmatrix} 1 \\ 1 \\ 0 \\ 1 \end{pmatrix} - \frac{0-2/3+1+1/3}{1/9+4/9+1+1/9}\begin{pmatrix} 1/3 \\ -2/3 \\ 1 \\ 1/3 \end{pmatrix}$$

$$= \begin{pmatrix} 0 \\ 1 \\ 1 \\ 1 \end{pmatrix} - \frac{2}{3}\begin{pmatrix} 1 \\ 1 \\ 0 \\ 1 \end{pmatrix} - \frac{2/3}{15/9}\begin{pmatrix} 1/3 \\ -2/3 \\ 1 \\ 1/3 \end{pmatrix} = \begin{pmatrix} -2/3 \\ 1/3 \\ 1 \\ 1/3 \end{pmatrix} - \frac{2}{5}\begin{pmatrix} 1/3 \\ -2/3 \\ 1 \\ 1/3 \end{pmatrix}$$

$$= \frac{1}{15}\begin{pmatrix} -10-2 \\ 5+4 \\ 15-6 \\ 5-2 \end{pmatrix} = \frac{1}{15}\begin{pmatrix} -12 \\ 9 \\ 9 \\ 3 \end{pmatrix} = \frac{1}{5}\begin{pmatrix} -4 \\ 3 \\ 3 \\ 1 \end{pmatrix}$$

$$\left\{\begin{pmatrix} 1 \\ 1 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ -2 \\ 3 \\ 1 \end{pmatrix}, \begin{pmatrix} -4 \\ 3 \\ 3 \\ 1 \end{pmatrix}\right\}$$

A.4. (20 pts.) Find the least squares regression line which best approximates the data points

$$(0,1),(1,0),(2,3),(3,5)$$

$$\boxed{y = ax+b}$$

$$A = \begin{pmatrix} 0 & 1 \\ 1 & 1 \\ 2 & 1 \\ 3 & 1 \end{pmatrix} \qquad \vec{y} = \begin{pmatrix} 1 \\ 0 \\ 3 \\ 5 \end{pmatrix}$$

$$A^TA = \begin{pmatrix} 0 & 1 & 2 & 3 \\ 1 & 1 & 1 & 1 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 1 \\ 2 & 1 \\ 3 & 1 \end{pmatrix} = \begin{pmatrix} 14 & 6 \\ 6 & 4 \end{pmatrix} \qquad \det(A^TA) = 56-36 = 20$$

$$A^T\vec{y} = \begin{pmatrix} 0 & 1 & 2 & 3 \\ 1 & 1 & 1 & 1 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \\ 3 \\ 5 \end{pmatrix} = \begin{pmatrix} 21 \\ 9 \end{pmatrix}$$

$$\begin{pmatrix} a \\ b \end{pmatrix} = (A^TA)^{-1}(A^T\vec{y}) = \frac{1}{20}\begin{pmatrix} 4 & -6 \\ -6 & 14 \end{pmatrix}\begin{pmatrix} 21 \\ 9 \end{pmatrix}$$

$$= \frac{3}{10}\begin{pmatrix} 2 & -3 \\ -3 & 7 \end{pmatrix}\begin{pmatrix} 7 \\ 3 \end{pmatrix} = \frac{3}{10}\begin{pmatrix} 5 \\ 0 \end{pmatrix} = \begin{pmatrix} 3/2 \\ 0 \end{pmatrix}$$

$$\boxed{y = 3/2\,x + 0}$$

$$\begin{pmatrix} 0 \\ 1 \\ 1 \\ 1 \end{pmatrix} - \frac{2}{3}\begin{pmatrix} 1 \\ 1 \\ 0 \\ 1 \end{pmatrix} - \frac{2}{5}\begin{pmatrix} 1/3 \\ -2/3 \\ 1 \\ 1/3 \end{pmatrix} = \frac{1}{15}\begin{pmatrix} 0-10-2 \\ 15-10+4 \\ 15-0-6 \\ 15-10-2 \end{pmatrix} = \frac{1}{15}\begin{pmatrix} -12 \\ 9 \\ 9 \\ 3 \end{pmatrix} = \frac{1}{5}\begin{pmatrix} -4 \\ 3 \\ 3 \\ 1 \end{pmatrix}$$

B.1. (15 pts.) Find the orthogonal projection of the vector $\vec{b} = \begin{bmatrix} 1 \\ 2 \\ 4 \end{bmatrix}$ onto the column space of the matrix

$$A = \begin{pmatrix} 2 & 2 \\ 3 & 1 \\ 1 & 1 \end{pmatrix}$$

$A\vec{x} = A(A^TA)^{-1}(A^T\vec{b})$

$A^T\vec{b} = \begin{pmatrix} 2 & 3 & 1 \\ 2 & 1 & 1 \end{pmatrix}\begin{pmatrix} 1 \\ 2 \\ 4 \end{pmatrix} = \begin{pmatrix} 12 \\ 8 \end{pmatrix}$

$A^TA = \begin{pmatrix} 2 & 3 & 1 \\ 2 & 1 & 1 \end{pmatrix}\begin{pmatrix} 2 & 2 \\ 3 & 1 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} 14 & 8 \\ 8 & 6 \end{pmatrix}$

$\det(A^TA) = 84 - 64 = \underline{20}$

$(A^TA)^{-1} = \frac{1}{20}\begin{pmatrix} 6 & -8 \\ -8 & 14 \end{pmatrix} = \frac{1}{10}\begin{pmatrix} 3 & -4 \\ -4 & 7 \end{pmatrix}$

$A\vec{x} = \begin{pmatrix} 2 & 2 \\ 3 & 1 \\ 1 & 1 \end{pmatrix}\frac{1}{10}\begin{pmatrix} 3 & -4 \\ -4 & 7 \end{pmatrix}\begin{pmatrix} 12 \\ 8 \end{pmatrix} = \begin{pmatrix} 2 & 2 \\ 3 & 1 \\ 1 & 1 \end{pmatrix}\frac{1}{10}\begin{pmatrix} 4 \\ 8 \end{pmatrix}$

$= \frac{1}{10}\begin{pmatrix} 2 & 2 \\ 3 & 1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} 4 \\ 8 \end{pmatrix} = \frac{1}{10}\begin{pmatrix} 24 \\ 20 \\ 12 \end{pmatrix} = \boxed{\begin{pmatrix} 12/5 \\ 2 \\ 6/5 \end{pmatrix}}$

or find basis for $W^\perp$ compute $\mathrm{proj}_{W^\perp}(\vec{b})$

then take $\vec{b} - \mathrm{proj}_{W^\perp}(\vec{b})$!

B.2. (15 pts.) If $T : \mathbb{R}^3 \to \mathbb{R}^2$ is a linear transformation with

$$T\begin{bmatrix}1\\2\\0\end{bmatrix} = \begin{bmatrix}1\\1\end{bmatrix} \quad , \quad T\begin{bmatrix}1\\1\\1\end{bmatrix} = \begin{bmatrix}3\\4\end{bmatrix} \quad , \quad \text{and } T\begin{bmatrix}0\\1\\2\end{bmatrix} = \begin{bmatrix}-1\\2\end{bmatrix} \quad ,$$

what is $T\begin{bmatrix}1\\2\\3\end{bmatrix}$?

$$\begin{pmatrix}1\\2\\3\end{pmatrix} = a\begin{pmatrix}1\\2\\0\end{pmatrix} + b\begin{pmatrix}1\\1\\1\end{pmatrix} + c\begin{pmatrix}0\\1\\2\end{pmatrix}$$

$$\left(\begin{array}{ccc|c}1&1&0&1\\2&1&1&2\\0&1&2&3\end{array}\right) \to \left(\begin{array}{ccc|c}1&1&0&1\\0&-1&1&0\\0&1&2&3\end{array}\right)$$

$$\to \left(\begin{array}{ccc|c}1&0&1&1\\0&1&-1&0\\0&0&3&3\end{array}\right) \to \left(\begin{array}{ccc|c}1&0&1&1\\0&1&-1&0\\0&0&1&1\end{array}\right) \to \left(\begin{array}{ccc|c}1&0&0&0\\0&1&0&1\\0&0&1&1\end{array}\right)$$

So $\begin{pmatrix}1\\2\\3\end{pmatrix} = 0\begin{pmatrix}1\\2\\0\end{pmatrix} + 1\begin{pmatrix}1\\1\\1\end{pmatrix} + 1\begin{pmatrix}0\\1\\2\end{pmatrix}$

So $T\begin{pmatrix}1\\2\\3\end{pmatrix} = 0\begin{pmatrix}1\\1\end{pmatrix} + 1\begin{pmatrix}3\\4\end{pmatrix} + 1\begin{pmatrix}-1\\2\end{pmatrix} = \underline{\underline{\begin{pmatrix}2\\6\end{pmatrix}}}$

$x = -a$
$y = 5$
$z = 7$

B.3. (15 pts.) For which values of x is the matrix $A = \begin{pmatrix} 2 & 1 & 3 \\ 1 & 2 & x \\ x & 1 & -1 \end{pmatrix}$ **not** invertible?

$$\det(A) = 2\begin{vmatrix} 2 & x \\ 1 & -1 \end{vmatrix} - 1\begin{vmatrix} 1 & x \\ x & -1 \end{vmatrix} + 3\begin{vmatrix} 1 & 2 \\ x & 1 \end{vmatrix}$$

$$= 2(-2-x) - 1(-1-x^2) + 3(1-2x)$$

$$= -4 - 2x + 1 + x^2 + 3 - 6x$$

$$= \cancel{x^2+4x-2}\ x^2 - 8x$$

A not invertible $\iff \det(A) = x^2 - 8x = 0$

$\iff x(x-8) = 0 \iff \boxed{x=0 \text{ or } x=8}$

B.4. (15 pts.) Suppose that A is an invertible $n \times n$ matrix. Show that if A is diagonalizable then A^{-1} is diagonalizable, as well.

If A is Δ-able then there is a P invertible and D diagonal so that

$$A = PDP^{-1}.$$ But then

$$A^{-1} = (PDP^{-1})^{-1} = (P^{-1})^{-1}(D)^{-1}(P)^{-1} = PD^{-1}P^{-1}$$

But P is still invertible, and

D^{-1} = a diagonal matrix

$$\begin{pmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{pmatrix}^{-1} = \begin{pmatrix} \lambda_1^{-1} & & 0 \\ & \ddots & \\ 0 & & \lambda_n^{-1} \end{pmatrix}$$ so A^{-1} is

diagonalizable. ///

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