Solutions

Name:

Math 314/814 Matrix Theory

Exam 1

Show all work. Include all steps necessary to arrive at an answer unaided by a mechanical computational device. The steps you take to your answer are just as important, if not more important, than the answer itself. If you think it, write it!

1. (20 pts.) Use row reduction to find a solution to the following system of linear equations:

$$\begin{aligned} 4x &+ 4y &+ 2z &= 3 \\ x &+ 3y &+ 2z &= 1 \\ 3x &+ 2y &+ z &= 1 \end{aligned}$$

$$\left(\begin{array}{ccc|c} 4 & 4 & 2 & 3 \\ 1 & 3 & 2 & 1 \\ 3 & 2 & 1 & 1 \end{array}\right) \longrightarrow \left(\begin{array}{ccc|c} 1 & 3 & 2 & 1 \\ 4 & 4 & 2 & 3 \\ 3 & 2 & 1 & 1 \end{array}\right) \rightarrow \left(\begin{array}{ccc|c} 1 & 3 & 2 & 1 \\ 0 & -8 & -6 & -1 \\ 0 & -7 & -5 & -2 \end{array}\right)$$

$$\rightarrow \left(\begin{array}{ccc|c} 1 & 3 & 2 & 1 \\ 0 & 1 & 3/4 & 1/8 \\ 0 & -7 & -5 & -2 \end{array}\right) \rightarrow \left(\begin{array}{ccc|c} 1 & 3 & 2 & 1 \\ 0 & 1 & 3/4 & 1/8 \\ 0 & 0 & 1/4 & -9/8 \end{array}\right) \rightarrow \left(\begin{array}{ccc|c} 1 & 3 & 2 & 1 \\ 0 & 1 & 3/4 & 1/8 \\ 0 & 0 & 1 & -9/2 \end{array}\right)$$

$$\rightarrow \left(\begin{array}{ccc|c} 1 & 3 & 0 & 10 \\ 0 & 1 & 0 & 28/8 \\ 0 & 0 & 1 & -9/2 \end{array}\right) \rightarrow \left(\begin{array}{ccc|c} 1 & 3 & 0 & 10 \\ 0 & 1 & 0 & 7/2 \\ 0 & 0 & 1 & -9/2 \end{array}\right) \rightarrow \left(\begin{array}{ccc|c} 1 & 0 & 0 & -1/2 \\ 0 & 1 & 0 & 7/2 \\ 0 & 0 & 1 & -9/2 \end{array}\right)$$

$$\boxed{x = -\frac{1}{2},\ y = \frac{7}{2},\ z = \frac{-9}{2}}$$

Check!

$$4\left(-\frac{1}{2}\right) + 4\left(\frac{7}{2}\right) + 2\left(\frac{-9}{2}\right) = -2 + 14 - 9 = 14 - 11 = 3 \checkmark$$

$$\left(\frac{-1}{2}\right) + 3\left(\frac{7}{2}\right) + 2\left(\frac{-9}{2}\right) = \frac{-1 + 21 - 18}{2} = \frac{21-19}{2} = \frac{2}{2} = 1 \checkmark$$

$$3\left(\frac{-1}{2}\right) + 2\left(\frac{7}{2}\right) + \left(\frac{-9}{2}\right) = \frac{-3 + 14 - 9}{2} = \frac{14-12}{2} = \frac{2}{2} = 1 \checkmark!$$

2. (20 pts.) A pet hotel can accept 50 dogs and 70 cats in its care. The average Belgian owns 2 dogs and 1 cat, while the average Luxembourgian owns 1 dog and 3 cats. How many Belgians and Luxembourgians can the pet hotel accomodate, if all of the available space is used?

x = # Belgians   y = # Luxembourgians

$50 = 2x + y$

$70 = x + 3y$

$$\left(\begin{array}{cc|c} 2 & 1 & 50 \\ 1 & 3 & 70 \end{array}\right) \rightarrow \left(\begin{array}{cc|c} 1 & 3 & 70 \\ 2 & 1 & 50 \end{array}\right) \rightarrow \left(\begin{array}{cc|c} 1 & 3 & 70 \\ 0 & -5 & -90 \end{array}\right)$$

$$\rightarrow \left(\begin{array}{cc|c} 1 & 3 & 70 \\ 0 & 1 & 18 \end{array}\right) \rightarrow \left(\begin{array}{cc|c} 1 & 0 & 70-54 \\ 0 & 1 & 18 \end{array}\right) \rightarrow \left(\begin{array}{cc|c} 1 & 0 & 16 \\ 0 & 1 & 18 \end{array}\right)$$

$x = 16$, $y = 18$

16 Belgians ad 18 Luxembourgians can be accomodated.

3. (25 pts.) Show that the vectors

$$\vec{v}_1 = \begin{bmatrix} 1 \\ 3 \\ 1 \end{bmatrix}, \vec{v}_2 = \begin{bmatrix} 2 \\ 5 \\ 3 \end{bmatrix}, \vec{v}_3 = \begin{bmatrix} 3 \\ 7 \\ 4 \end{bmatrix}, \vec{v}_4 = \begin{bmatrix} 4 \\ 3 \\ 2 \end{bmatrix}$$

are linearly dependent, and exhibit an explicit linear dependence among them.

$$\begin{array}{cccccc} x & y & z & w & & \\ \end{array}$$

$$\left(\begin{array}{cccc|c} 1 & 2 & 3 & 4 & 0 \\ 3 & 5 & 7 & 3 & 0 \\ 1 & 3 & 4 & 2 & 0 \end{array}\right) \to \left(\begin{array}{cccc|c} 1 & 2 & 3 & 4 & 0 \\ 0 & -1 & -2 & -9 & 0 \\ 0 & 1 & 1 & -2 & 0 \end{array}\right)$$

$$\to \left(\begin{array}{cccc|c} 1 & 2 & 3 & 4 & 0 \\ 0 & 1 & 1 & -2 & 0 \\ 0 & 1 & 2 & 9 & 0 \end{array}\right) \to \left(\begin{array}{cccc|c} 1 & 2 & 3 & 4 & 0 \\ 0 & 1 & 1 & -2 & 0 \\ 0 & 0 & 1 & 11 & 0 \end{array}\right)$$

$$\to \left(\begin{array}{cccc|c} 1 & 2 & 0 & -29 & 0 \\ 0 & 1 & 0 & -13 & 0 \\ 0 & 0 & 1 & 11 & 0 \end{array}\right) \to \left(\begin{array}{cccc|c} 1 & 0 & 0 & -3 & 0 \\ 0 & 1 & 0 & -13 & 0 \\ 0 & 0 & 1 & 11 & 0 \end{array}\right)$$

$x - 3w = 0$
$y - 13w = 0$
$z + 11w = 0$

$x = 3w$, $y = 13w$, $z = -11w$ So

Set $w = 1$! $\to$ $x = 3$, $y = 13$, $z = -11$

$$3\vec{v}_1 + 13\vec{v}_2 - 11\vec{v}_3 + v_4 = \vec{0}:$$

Check!

$$3\begin{pmatrix} 1 \\ 3 \\ 1 \end{pmatrix} + 13\begin{pmatrix} 2 \\ 5 \\ 3 \end{pmatrix} - 11\begin{pmatrix} 3 \\ 7 \\ 4 \end{pmatrix} + \begin{pmatrix} 4 \\ 3 \\ 2 \end{pmatrix} = \begin{pmatrix} 3+26-33+4 \\ 9+65-77+3 \\ 3+39-44+2 \end{pmatrix}$$

$$= \begin{pmatrix} 33-33 \\ 77-77 \\ 44-44 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \checkmark!$$

4. (25 pts.) Use row reduction to find the values of x for which the following matrix is **invertible:**

$$\mathrm{A(x)} = \begin{pmatrix} 1 & 2 & 3 \\ 3 & x & -1 \\ 2 & 2 & -x \end{pmatrix}$$

$$\begin{pmatrix} 1 & 2 & 3 \\ 3 & x & -1 \\ 2 & 2 & -x \end{pmatrix} \to \begin{pmatrix} 1 & 2 & 3 \\ 0 & x-6 & -10 \\ 0 & -2 & -x-6 \end{pmatrix} \to \begin{pmatrix} 1 & 2 & 3 \\ 0 & -2 & -x-6 \\ 0 & x-6 & -10 \end{pmatrix}$$

$$\to \begin{pmatrix} 1 & 2 & 3 \\ 0 & 1 & \frac{x+6}{2} \\ 0 & x-6 & -10 \end{pmatrix} \to \begin{pmatrix} 1 & 2 & 3 \\ 0 & 1 & \frac{x+6}{2} \\ 0 & 0 & -10-(x-6)\left(\frac{x+6}{2}\right) \end{pmatrix}$$

This is not invertible only if $-10-(x-6)\left(\frac{x+6}{2}\right)=0$

(Otherwise, we have 3 pivots, so we can continue to row reduce to $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ $\leadsto$ invertible!)

$$-10-(x-6)\left(\frac{x+6}{2}\right)=0 \iff 10+(x-6)\left(\frac{x+6}{2}\right)=0$$

$$\iff 20+(x-6)(x+6) = 20+x^2-36 = x^2-16=0$$

$$\iff x^2=16 \iff x=-4,4$$

So for $x \neq -4, 4$, $A(x)$ is invertible!

5. (10 pts.) An $n \times n$ matrix A is called **nilpotent** if $A^k = 0_{n\times n}$ for some number k. Show that if A is nilpotent, then $I - A$ is an **invertible** matrix. [Hint: "factor" $I = I - A^k = (I - A)(\text{what?})$. You can try $k = 2, 3$, or 4 first to give you some feel for the general case...]

Since $A^k = \underline{0}$,

$$I = I - A^k = (I-A)(I + A + A^2 + \cdots + A^{k-1})$$

Setting $B = I + A + A^2 + \cdots + A^{k-1}$,

$$(I-A)B = I.$$

Since all of the matrices are square, this implies that $B = (I-A)^{-1}$

(or check directly!
$$(I + A + \cdots + A^{k-1})(I-A) = I + \cancel{A} + \cdots + \cancel{A^{k-1}} - (\cancel{A} + \cancel{A^2} + \cdots + A^k)$$
$$= I - A^k = I - \underline{0} = I!$$
)

So $(I-A)$ has an inverse! So it is invertible.

In steps:

$$I - A^2 = I - A \cdot A = (I-A)(I+A)$$
$$= I - A + A(I-A)A$$
$$= I - A + IA - AA = I - \cancel{A} + \cancel{A} - A^2$$

$$I - A^3 = I - AAA = (I-A)(I + A + A^2)$$
$$= I - A + (I-A)A + (I-A)A^2$$
$$= I - \cancel{A} + \cancel{IA} - \cancel{A^2} + \cancel{IA^2} - A^3$$

$$I - A^4 = (I-A)(I + A + A^2 + A^3)$$