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Name: Solutions

Math 314 Exam 1

Show all work. Include all steps necessary to arrive at an answer unaided by a mechanical computational device. The steps you take to your answer are just as important, if not more important, than the answer itself. If you think it, write it!

1. (20 pts.) A dollar can buy

2 apples and 1 pear, or it can buy
1 apple, 2 pears and 2 grapes, or it can buy
1 apple, 1 pear, and 6 grapes.

How many pears does a dollar buy?

want prices p_a, p_p, p_g

Really want p_p $!

$$2p_a + 1p_p = 1$$
$$1p_a + 2p_p + 2p_g = 1$$
$$1p_a + 1p_p + 6p_g = 1$$

$$\left(\begin{array}{ccc|c} 2 & 1 & 0 & 1 \\ 1 & 2 & 2 & 1 \\ 1 & 1 & 6 & 1 \end{array}\right) \longrightarrow \left(\begin{array}{ccc|c} 1 & 1 & 6 & 1 \\ 1 & 2 & 2 & 1 \\ 2 & 1 & 0 & 1 \end{array}\right) \longrightarrow \left(\begin{array}{ccc|c} 1 & 1 & 6 & 1 \\ 0 & 1 & -4 & 0 \\ 0 & -1 & -12 & -1 \end{array}\right)$$

$$\longrightarrow \left(\begin{array}{ccc|c} 1 & 1 & 6 & 1 \\ 0 & 1 & -4 & 0 \\ 0 & 0 & -16 & -1 \end{array}\right) \longrightarrow \left(\begin{array}{ccc|c} 1 & 1 & 6 & 1 \\ 0 & 1 & -4 & 0 \\ 0 & 0 & 1 & 1/16 \end{array}\right) \longrightarrow \left(\begin{array}{ccc|c} 1 & 1 & 6 & 1 \\ 0 & 1 & 0 & 1/4 \\ 0 & 0 & 1 & 1/16 \end{array}\right)$$

$$\longrightarrow \left(\begin{array}{ccc|c} 1 & 1 & 0 & 5/8 \\ 0 & 1 & 0 & 1/4 \\ 0 & 0 & 1 & 1/16 \end{array}\right) \longrightarrow \left(\begin{array}{ccc|c} 1 & 0 & 0 & 3/8 \\ 0 & 1 & 0 & 1/4 \\ 0 & 0 & 1 & 1/16 \end{array}\right)$$

One pear costs $\frac{1}{4}$ of a dollar, so $\boxed{\text{1 dollar buys } \underline{4} \text{ pears.}}$

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2. (20 pts.) For which value(s) of x does the system of linear equations, given by the augmented matrix

$$A = \begin{pmatrix} 2 & 1 & 1 & | & 2 \\ 1 & -1 & 2 & | & 1 \\ 3 & 1 & x & | & 1 \end{pmatrix},$$

have **<u>no</u>** solution?

Row reduce!

$$\left(\begin{array}{ccc|c} 2 & 1 & 1 & 2 \\ 1 & -1 & 2 & 1 \\ 3 & 1 & x & 1 \end{array}\right) \longrightarrow \left(\begin{array}{ccc|c} 1 & -1 & 2 & 1 \\ 2 & 1 & 1 & 2 \\ 3 & 1 & x & 1 \end{array}\right) \longrightarrow \left(\begin{array}{ccc|c} 1 & -1 & 2 & 1 \\ 0 & 3 & -3 & 0 \\ 3 & 1 & x & 1 \end{array}\right)$$

$$\longrightarrow \left(\begin{array}{ccc|c} 1 & -1 & 2 & 1 \\ 0 & 3 & -3 & 0 \\ 0 & 4 & x-6 & -2 \end{array}\right) \longrightarrow \left(\begin{array}{ccc|c} 1 & -1 & 2 & 1 \\ 0 & 1 & -1 & 0 \\ 0 & 4 & x-6 & -2 \end{array}\right) \longrightarrow \left(\begin{array}{ccc|c} 1 & -1 & 2 & 1 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & x-2 & -2 \end{array}\right)$$

$x-2$ is a <u>pivot</u> (so we <u>have</u> a solution) <u>unless</u> $x-2=0$ (i.e., $x=2$)

If $x=2$ then last row is $(0\ 0\ 0 \mid -2) \rightarrow$ <u>inconsistent</u>

So the SLE has <u>no</u> solution precisely when $\boxed{x=2}$.

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3. (20 pts.) Are the vectors $\begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}$, $\begin{bmatrix} 2 \\ 2 \\ -3 \end{bmatrix}$, and $\begin{bmatrix} 3 \\ 1 \\ 5 \end{bmatrix}$ linearly independent?

$\vec{u}$ $\vec{v}$ $\vec{w}$

Does $a\vec{u} + b\vec{v} + c\vec{w} = \vec{0}$ only when $a=b=c=0$?

Solve $\left(\begin{array}{ccc|c} 1 & 2 & 3 & 0 \\ 2 & 2 & 1 & 0 \\ -1 & -3 & 5 & 0 \end{array}\right)$! Row reduce!

$$\to \left(\begin{array}{ccc|c} 1 & 2 & 3 & 0 \\ 0 & -2 & -5 & 0 \\ 0 & -1 & 8 & 0 \end{array}\right) \to \left(\begin{array}{ccc|c} 1 & 2 & 3 & 0 \\ 0 & 1 & -8 & 0 \\ 0 & -2 & -5 & 0 \end{array}\right)$$

$$\to \left(\begin{array}{ccc|c} 1 & 2 & 3 & 0 \\ 0 & 1 & -8 & 0 \\ 0 & 0 & -21 & 0 \end{array}\right) \to \left(\begin{array}{ccc|c} 1 & 2 & 3 & 0 \\ 0 & 1 & -8 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right) \to \left(\begin{array}{ccc|c} 1 & 2 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right)$$

$$\to \left(\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right) \to a=b=c=0.$$

So only solution is the all-0 solution, so $\vec{u}, \vec{v}, \vec{w}$ are linearly independent.

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4. (15 pts.) The linear transformation $T : \mathbf{R}^2 \to \mathbf{R}^2$ has

$$T\begin{bmatrix}2\\1\end{bmatrix} = \begin{bmatrix}3\\2\end{bmatrix} \text{ and } T\begin{bmatrix}1\\1\end{bmatrix} = \begin{bmatrix}0\\1\end{bmatrix}.$$

What is $T\begin{bmatrix}-1\\1\end{bmatrix}$?

(Hint: how can you express $\begin{bmatrix}-1\\1\end{bmatrix}$ in terms of $\begin{bmatrix}2\\1\end{bmatrix}$ and $\begin{bmatrix}1\\1\end{bmatrix}$?)

If $\begin{bmatrix}-1\\1\end{bmatrix} = a\begin{bmatrix}2\\1\end{bmatrix} + b\begin{bmatrix}1\\1\end{bmatrix}$, then

$$T\begin{pmatrix}-1\\1\end{pmatrix} = T\left(a\begin{pmatrix}2\\1\end{pmatrix} + b\begin{pmatrix}1\\1\end{pmatrix}\right) = aT\begin{pmatrix}2\\1\end{pmatrix} + bT\begin{pmatrix}1\\1\end{pmatrix}$$

$$= a\begin{pmatrix}3\\2\end{pmatrix} + b\begin{pmatrix}0\\1\end{pmatrix}.$$

But asks us to solve the SLE

$$\left(\begin{array}{cc|c}2 & 1 & -1\\1 & 1 & 1\end{array}\right) \to \left(\begin{array}{cc|c}1 & 1 & 1\\2 & 1 & -1\end{array}\right) \to \left(\begin{array}{cc|c}1 & 1 & 1\\0 & -1 & -3\end{array}\right)$$

$$\to \left(\begin{array}{cc|c}1 & 1 & 1\\0 & 1 & 3\end{array}\right) \to \left(\begin{array}{cc|c}1 & 0 & -2\\0 & 1 & 3\end{array}\right), \text{ so } a = -2, b = 3$$

[Check: $-2\begin{pmatrix}2\\1\end{pmatrix} + 3\begin{pmatrix}1\\1\end{pmatrix} = \begin{pmatrix}-4+3\\-2+3\end{pmatrix} = \begin{pmatrix}-1\\1\end{pmatrix}$ ✓]

so $T\begin{pmatrix}-1\\1\end{pmatrix} = (-2)\begin{pmatrix}3\\2\end{pmatrix} + 3\begin{pmatrix}0\\1\end{pmatrix} = \begin{pmatrix}-6+0\\-4+3\end{pmatrix} = \boxed{\begin{pmatrix}-6\\-1\end{pmatrix}}$

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5. (25 pts.) Find the inverse of the matrix

$$A = \begin{pmatrix} 1 & 3 & 2 \\ 2 & 3 & 3 \\ 1 & 5 & 3 \end{pmatrix},$$

and use this inverse to find solutions to the systems of equations $A\vec{x} = \vec{b}$, for

$$\vec{b} = \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix} \text{ and } \vec{b} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}.$$

Super-augmented matrix:

$$\left(\begin{array}{ccc|ccc} 1 & 3 & 2 & 1 & 0 & 0 \\ 2 & 3 & 3 & 0 & 1 & 0 \\ 1 & 5 & 3 & 0 & 0 & 1 \end{array}\right) \to \left(\begin{array}{ccc|ccc} 1 & 3 & 2 & 1 & 0 & 0 \\ 0 & -3 & -1 & -2 & 1 & 0 \\ 1 & 5 & 3 & 0 & 0 & 1 \end{array}\right) \to \left(\begin{array}{ccc|ccc} 1 & 3 & 2 & 1 & 0 & 0 \\ 0 & -3 & -1 & -2 & 1 & 0 \\ 0 & 2 & 1 & -1 & 0 & 1 \end{array}\right)$$

$$\to \left(\begin{array}{ccc|ccc} 1 & 3 & 2 & 1 & 0 & 0 \\ 0 & 1 & 1/3 & 2/3 & -1/3 & 0 \\ 0 & 2 & 1 & -1 & 0 & 1 \end{array}\right) \to \left(\begin{array}{ccc|ccc} 1 & 3 & 2 & 1 & 0 & 0 \\ 0 & 1 & 1/3 & 2/3 & -1/3 & 0 \\ 0 & 0 & 1/3 & -7/3 & 2/3 & 1 \end{array}\right) \to \left(\begin{array}{ccc|ccc} 1 & 3 & 2 & 1 & 0 & 0 \\ 0 & 1 & 1/3 & 2/3 & -1/3 & 0 \\ 0 & 0 & 1 & -7 & 2 & 3 \end{array}\right)$$

$$\to \left(\begin{array}{ccc|ccc} 1 & 3 & 2 & 1 & 0 & 0 \\ 0 & 1 & 0 & 3 & -1 & -1 \\ 0 & 0 & 1 & -7 & 2 & 3 \end{array}\right) \to \left(\begin{array}{ccc|ccc} 1 & 3 & 0 & 15 & -4 & -6 \\ 0 & 1 & 0 & 3 & -1 & -1 \\ 0 & 0 & 1 & -7 & 2 & 3 \end{array}\right) \to \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 6 & -1 & -3 \\ 0 & 1 & 0 & 3 & -1 & -1 \\ 0 & 0 & 1 & -7 & 2 & 3 \end{array}\right)$$

So $A^{-1} = \begin{pmatrix} 6 & -1 & -3 \\ 3 & -1 & -1 \\ -7 & 2 & 3 \end{pmatrix}$ Then $A\vec{x} = \vec{b}$ has solution $\vec{x} = A^{-1}\vec{b}$,

So our solutions are

$$\vec{x} = A^{-1}\begin{pmatrix} 3 \\ 2 \\ 1 \end{pmatrix} = \begin{pmatrix} 6 & -1 & -3 \\ 3 & -1 & -1 \\ -7 & 2 & 3 \end{pmatrix}\begin{pmatrix} 3 \\ 2 \\ 1 \end{pmatrix} = \begin{pmatrix} 18 - 2 - 3 \\ 9 - 2 - 1 \\ -21 + 4 + 3 \end{pmatrix} = \boxed{\begin{pmatrix} 13 \\ 6 \\ -14 \end{pmatrix}}$$

and

$$\vec{x} = A^{-1}\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} = \begin{pmatrix} 6 & -1 & -3 \\ 3 & -1 & -1 \\ -7 & 2 & 3 \end{pmatrix}\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} = \begin{pmatrix} 6 - 2 - 9 \\ 3 - 2 - 3 \\ -7 + 4 + 9 \end{pmatrix} = \boxed{\begin{pmatrix} -5 \\ -2 \\ 6 \end{pmatrix}}$$

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Check!

$$\begin{pmatrix} 1 & 3 & 2 \\ 2 & 3 & 3 \\ 1 & 5 & 3 \end{pmatrix}\begin{pmatrix} 13 \\ 6 \\ -14 \end{pmatrix} = \begin{pmatrix} 13+18-28 \\ 26+18-42 \\ 13+30-42 \end{pmatrix} = \begin{pmatrix} 31-28 \\ 44-42 \\ 43-42 \end{pmatrix} = \begin{pmatrix} 3 \\ 2 \\ 1 \end{pmatrix} \checkmark$$

$$\begin{pmatrix} 1 & 3 & 2 \\ 2 & 3 & 3 \\ 1 & 5 & 3 \end{pmatrix}\begin{pmatrix} -5 \\ -2 \\ 6 \end{pmatrix} = \begin{pmatrix} -5-6+12 \\ -10-6+18 \\ -5-10+18 \end{pmatrix} = \begin{pmatrix} 12-11 \\ 18-16 \\ 18-15 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} \checkmark$$

Alternate approach to Problem #4: Find the matrix for T!

$$T\begin{pmatrix} 2 \\ 1 \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} 2 \\ 1 \end{pmatrix} = \begin{pmatrix} 2a+b \\ 2c+d \end{pmatrix} = \begin{pmatrix} 3 \\ 2 \end{pmatrix}$$

$$T\begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} a+b \\ c+d \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

$2a+b=3$
$2c+d=2$
$a+b=0$
$c+d=1$

$a+b=0,\ 2a+b=3 \longrightarrow a=(2a+b)-(a+b)=3-0=3$
$b=0-a=-3$

$c+d=1,\ 2c+d=2 \longrightarrow c=(2c+d)-(c+d)=2-1=1$
$d=1-c=1-1=0$

So $T=T_A$, for $A=\begin{pmatrix} 3 & -3 \\ 1 & 0 \end{pmatrix}$. So

$$T\begin{pmatrix} -1 \\ 1 \end{pmatrix} = T_A\begin{pmatrix} -1 \\ 1 \end{pmatrix} = \begin{pmatrix} 3 & -3 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} -1 \\ 1 \end{pmatrix} = \begin{pmatrix} -3-3 \\ -1+0 \end{pmatrix} = \boxed{\begin{pmatrix} -6 \\ -1 \end{pmatrix}}.$$