# Math 314

# Exam 2 practice problems

# Solutions

Solutions

**Name:**

## Math 314 Matrix Theory

## Exam 2

Show all work. How you get your answer is just as important, if not more important, than the answer itself. If you think it, write it!

**1.** (20 pts.) Find, using any method (other than psychic powers), the determinant of the matrix

$$A = \begin{pmatrix} 1 & 2 & 1 \\ 0 & -1 & 3 \\ 4 & 4 & 2 \end{pmatrix}$$

Is this matrix invertible?

$$\longrightarrow \begin{pmatrix} 1 & 2 & 1 \\ 0 & -1 & 3 \\ 0 & -4 & -2 \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & 2 & 1 \\ 0 & -1 & 3 \\ 0 & 0 & -14 \end{pmatrix}$$

$$\det(A) = 1 \cdot (-1)(-14) = 14$$

$\det(A) \neq 0$ so $A$ is invertible

or

$$\det(A) = 1 \begin{vmatrix} -1 & 3 \\ 4 & 2 \end{vmatrix} - 0 \begin{vmatrix} 2 & 1 \\ 4 & 2 \end{vmatrix} + 4 \begin{vmatrix} 2 & 1 \\ -1 & 3 \end{vmatrix}$$

$$= 1(-14) + 4(7) = 14.$$

**2.** (15 pts.) Explain why the set of vectors

$$W = \{(x, y, z) \mid x + y + 2z = 1\}$$

is **not** a subspace of $\mathbf{R}^3$.

How many reasons do you want?

$0+0+2(0) = 0 \neq 1$ so $(0,0,0) \notin W$
so it can't be a subspace.

$\overset{u}{(1,0,0)}, \overset{v}{(0,1,0)} \in W$ but:

(a) $u+v = (1,1,0)$ has $1+1+2(0) = 2 \neq 1$ so
$u+v \notin W$, so it can't be...

(b) $2 \cdot u = (2,0,0)$ has $2+0+2(0) = 2 \neq 1$
so $2u \notin W$ so it can't be...

$-u = (-1,0,0)$ has $(-1)+0+2(0) = -1 \neq 1$ so it
can't be....

[Any one answer will do!]

**4.**(20 pts.) For the matrix

$$A = \begin{pmatrix} 1 & 2 & 1 & 0 \\ 2 & 1 & 3 & 2 \\ 1 & 1 & 1 & 1 \end{pmatrix}$$

find bases for, and the dimensions of, the row, column, and null spaces of $A$.

$$A \longrightarrow \begin{pmatrix} 1 & 2 & 1 & 0 \\ 0 & -3 & 1 & 2 \\ 0 & -1 & 0 & 1 \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & 2 & 1 & 0 \\ 0 & 1 & 0 & -1 \\ 0 & -3 & 1 & 2 \end{pmatrix}$$

$$\longrightarrow \begin{pmatrix} 1 & 2 & 1 & 0 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & -1 \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & 0 & 1 & 2 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & -1 \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & 0 & 0 & 3 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & -1 \end{pmatrix}$$

$\mathcal{R}(A)$: basis $= \begin{pmatrix} 1 \\ 0 \\ 0 \\ 3 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \\ -1 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \\ -1 \end{pmatrix}$ dim $= \underline{3}$

$\mathcal{C}(A)$: basis $= \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}, \begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 3 \\ 1 \end{pmatrix}$ dim $= \underline{3}$

$\mathcal{N}(A)$: $\begin{matrix} x_1 + 3x_4 = 0 \\ x_2 - x_4 = 0 \\ x_3 - x_4 = 0 \end{matrix}$ $\quad \begin{matrix} x_1 = -3x_4 \\ x_2 = x_4 \\ x_3 = x_4 \end{matrix}$ $\quad \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = x_4 \underbrace{\begin{pmatrix} -3 \\ 1 \\ 1 \\ 1 \end{pmatrix}}_{\text{basis}}$ dim $= \underline{1}$

**5.** (20 pts.) Find **all** of the solutions to the equation $Ax = b$, where

$$A = \begin{pmatrix} 1 & 2 & 2 & 1 \\ 2 & 4 & 3 & 3 \\ 1 & 2 & 1 & 2 \end{pmatrix} \text{ and } b = \begin{pmatrix} -2 \\ -2 \\ 0 \end{pmatrix}$$

$$\left(\begin{array}{cccc|c} 1 & 2 & 2 & 1 & -2 \\ 2 & 4 & 3 & 3 & -2 \\ 1 & 2 & 1 & 2 & 0 \end{array}\right) \longrightarrow \left(\begin{array}{cccc|c} 1 & 2 & 2 & 1 & -2 \\ 0 & 0 & -1 & 1 & 2 \\ 0 & 0 & -1 & +1 & 2 \end{array}\right)$$

$$\longrightarrow \left(\begin{array}{cccc|c} 1 & 2 & 2 & 1 & -2 \\ 0 & 0 & 1 & -1 & -2 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right) \longrightarrow \left(\begin{array}{cccc|c} 1 & 2 & 0 & 3 & 2 \\ 0 & 0 & 1 & -1 & -2 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right)$$

$$x_1 + 2x_2 + 3x_4 = 2 \qquad x_1 = 2 - 2x_2 - 3x_4$$

$$x_3 - x_4 = -2 \qquad x_3 = -2 + x_4$$

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 2 \\ 0 \\ -2 \\ 0 \end{pmatrix} + x_2 \begin{pmatrix} -2 \\ 1 \\ 0 \\ 0 \end{pmatrix} + x_4 \begin{pmatrix} -3 \\ 0 \\ 1 \\ 1 \end{pmatrix}$$

**5.** A friend of yours runs up to you and says 'Look I've found these three vectors $v_1, v_2, v_3$ in $\mathbf{R}^2$ that are linearly independent!' Explain how you know, without even looking at the vectors, that your friend is wrong (again).

3 vectors in $\mathbb{R}^2$ can't be linearly independent, because if we write them as columns of a matrix and row reduce $(v_1 v_2 v_3) = A \longrightarrow R$

$R$ ~~can~~ has pivots in different rows, so has at most 2 pivots. Since $R$ has 3 columns, it therefore has a free variable, so $A\vec{x} = \vec{0}$ has a non-$\vec{0}$ solution. This gives a non-trivial linear combination $av_1 + bv_2 + cv_3 = \vec{0}$, so the vectors are linearly dependent!

Solutions

**Name:**

**M314 Matrix Theory**
**Exam 2**

Exams provide you the student with an opportunity to demonstrate your understanding of the techniques presented in the course. So:

**Show all work.** The steps you take to your answer are just as important, if not more important, than the answer itself. If you think it, write it!

**1.** (20 pts.) Find, using any method (other than psychic powers), the determinant of the matrix

$$A = \begin{pmatrix} -1 & 0 & 1 & 2 \\ 3 & -3 & 1 & 1 \\ 0 & 2 & 0 & 3 \\ 1 & -1 & 1 & 1 \end{pmatrix}$$

Is this matrix invertible?

$$|A| = (-1)\begin{vmatrix} 1 & 0 & -1 & -2 \\ 0 & -3 & 4 & 7 \\ 0 & 2 & 0 & 3 \\ 0 & -1 & 2 & 3 \end{vmatrix} \Rightarrow (-1)^3\begin{vmatrix} 1 & 0 & -1 & -2 \\ 0 & 1 & -2 & -3 \\ 0 & 2 & 0 & 3 \\ 0 & -3 & 4 & 7 \end{vmatrix} = (-1)^3\begin{vmatrix} 1 & 0 & -1 & -2 \\ 0 & 1 & -2 & -3 \\ 0 & 0 & 4 & 9 \\ 0 & 0 & -2 & -2 \end{vmatrix}$$

$$= (-1)^3(-1)(-2)\begin{vmatrix} 1 & 0 & -1 & -2 \\ 0 & 1 & -2 & -3 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 4 & 9 \end{vmatrix} = (-1)^3(-1)(-2)\begin{vmatrix} 1 & 0 & -1 & -2 \\ 0 & 1 & -2 & -3 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 5 \end{vmatrix}$$

$$= (-2)(1)(1)(1)(5) = -10 \neq 0 \quad \therefore A \text{ is invertible.}$$

or (expand on 3rd row)

$$\det|A| = 0\begin{vmatrix} \cancel{-1} & 0 & 1 & 2 \\ \cancel{3} & -3 & 1 & 1 \\ \cancel{1} & -1 & 1 & 1 \end{vmatrix} - 2\begin{vmatrix} -1 & 1 & 2 \\ 3 & 1 & 1 \\ 1 & 1 & 1 \end{vmatrix} + 0\begin{vmatrix} -1 & 0 & 2 \\ 3 & -3 & 1 \\ 1 & 1 & 1 \end{vmatrix} - 3\begin{vmatrix} -1 & 0 & 1 \\ 3 & -3 & 1 \\ 1 & -1 & 1 \end{vmatrix}$$

$$= -2\left(-1\begin{vmatrix} 1 & 1 \\ 1 & 1 \end{vmatrix} - 1\begin{vmatrix} 3 & 1 \\ 1 & 1 \end{vmatrix} + 2\begin{vmatrix} 3 & 1 \\ 1 & 1 \end{vmatrix}\right) - 3\left(-1\begin{vmatrix} -3 & 1 \\ -1 & 1 \end{vmatrix} - 0\begin{vmatrix} 3 & 1 \\ 1 & 1 \end{vmatrix} + 1\begin{vmatrix} 3 & -3 \\ -1 & 1 \end{vmatrix}\right)$$

$$= -2(0 - 2 + 4) - 3(\overset{2}{\cancel{4}} - 0 + \overset{0}{\cancel{6}})$$

$$= -4 - \cancel{30}\,6 = -10 \quad //$$

or [expand on any other row or column...]

**3.** The system of equations

$$\left(\underbrace{\begin{matrix}1&1&1&1\\2&2&3&4\\3&3&-1&-6\\1&1&2&3\end{matrix}}_{A}\;\middle|\;\underbrace{\begin{matrix}1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&1\end{matrix}}_{I}\right)\text{ row-reduces to }\left(\underbrace{\begin{matrix}1&1&0&0\\0&0&1&0\\0&0&0&1\\0&0&0&0\end{matrix}}_{R}\;\middle|\;\underbrace{\begin{matrix}14&-5&-1&0\\-24&9&2&0\\11&-4&-1&0\\1&-1&0&1\end{matrix}}_{Q}\right).$$

If we call the left-hand side of the first pair of matrices A, use this row-reduction information to find the dimensions and bases for the subspaces Row(A), Nul(A), and Row($A^T$).

(5 pts. for each subspace.)

Row(A) has basis (the transposes of) the non-0 rows of R, so $\begin{pmatrix}1\\1\\0\\0\end{pmatrix}, \begin{pmatrix}0\\0\\1\\0\end{pmatrix}, \begin{pmatrix}0\\0\\0\\1\end{pmatrix}$ are a basis for Row(A)

R has one free variable, so Nul(A) has one basis vector. $y$ is free.

$$\begin{aligned}x+y&=0\\z&=0\\w&=0\end{aligned} \quad \text{gives} \quad \begin{aligned}x&=-y\\y&=y\\z&=0\\w&=0\end{aligned} \qquad \vec{x} = y\begin{pmatrix}-1\\1\\0\\0\end{pmatrix} \text{ so}$$

$\begin{pmatrix}-1\\1\\0\\0\end{pmatrix}$ is a basis for Nul(A)

Row($A^T$) = Col(A), and Col(A) has basis the pivot columns of A. Pivots are in columns 1, 2, and 4, so $\begin{pmatrix}1\\2\\3\\1\end{pmatrix}, \begin{pmatrix}1\\3\\-1\\2\end{pmatrix}, \begin{pmatrix}1\\4\\-6\\3\end{pmatrix}$ is a basis for Row($A^T$).

**3.** Do the vectors $\begin{pmatrix}1\\-1\\3\end{pmatrix}$, $\begin{pmatrix}2\\0\\2\end{pmatrix}$, $\begin{pmatrix}-1\\1\\1\end{pmatrix}$, and $\begin{pmatrix}1\\3\\5\end{pmatrix}$ span $\mathbf{R}^3$?

Are they linearly independent?
Can you find a subset of this collection of vectors which forms a basis for $\mathrm{R}^3$?
(10 pts. for spanning, 10 pts. for lin indep, 5 pts. for basis.)

Both of the first 2 questions can be answered by row-reducing

$$\begin{pmatrix}1 & 2 & -1 & 1\\ -1 & 0 & 1 & 3\\ 3 & 2 & 1 & 5\end{pmatrix} \longrightarrow \begin{pmatrix}1 & 2 & -1 & 1\\ 0 & 2 & 0 & 4\\ 0 & -4 & 4 & 2\end{pmatrix}$$

$$\longrightarrow \begin{pmatrix}1 & 2 & -1 & 1\\ 0 & 1 & 0 & \cancel{8}2\\ 0 & -4 & 4 & 2\end{pmatrix} \longrightarrow \begin{pmatrix}1 & 2 & -1 & 1\\ 0 & 1 & 0 & 2\\ 0 & 0 & 4 & 10\end{pmatrix} \quad \text{REF}.$$

we have 3 pivots, so we have a pivot in every row, so they <u>span</u> $\mathbb{R}^3$. We have a free variable so they are <u>not</u> lin indep. But if we use only the first 3 vectors, then we have no free var, and we still have a pivot in each row, so $\begin{pmatrix}1\\-1\\3\end{pmatrix}, \begin{pmatrix}2\\0\\2\end{pmatrix}, \begin{pmatrix}-1\\1\\1\end{pmatrix}$ both span and are lin indep, so they are a <u>basis</u> for $\mathbb{R}^3$.

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Solutions

**Name:**

# Math 314 Matrix Theory
## Exam 2

Show all work. How you get your answer is just as important, if not more important, than the answer itself. If you think it, write it!

**1.** (20 pts.) Find the determinant of the matrix

$$A = \begin{pmatrix} 1 & 2 & 1 & -1 \\ 3 & -1 & 2 & 3 \\ 5 & 4 & 2 & 1 \\ 2 & 4 & 2 & -3 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 2 & 1 & -1 \\ 3 & -1 & 2 & 3 \\ 5 & 4 & 2 & 1 \\ 2 & 4 & 2 & -3 \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & 2 & 1 & -1 \\ 0 & -7 & -1 & 6 \\ 0 & -6 & -3 & 6 \\ 0 & 0 & 0 & -1 \end{pmatrix} \xrightarrow{-1} \begin{pmatrix} 1 & 2 & 1 & -1 \\ 0 & -6 & -3 & 6 \\ 0 & -7 & -1 & 6 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

$$\xrightarrow{-\frac{1}{6}} \begin{pmatrix} 1 & 2 & 1 & -1 \\ 0 & 1 & 1/2 & -1 \\ 0 & -7 & -1 & 6 \\ 0 & 0 & 0 & -1 \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & 2 & 1 & -1 \\ 0 & 1 & 1/2 & -1 \\ 0 & 0 & 5/2 & -1 \\ 0 & 0 & 0 & -1 \end{pmatrix} = R$$

$$\det(R) = (1)(1)\left(\tfrac{5}{2}\right)(-1) = (-1)\left(-\tfrac{1}{6}\right)\det(A) \text{, so}$$

$$\det(A) = (-1)(-6)(1)(1)\left(\tfrac{5}{2}\right)(-1) = 6\left(\tfrac{-5}{2}\right) = \left(\tfrac{-30}{2}\right) = \boxed{-15}$$

**2.** (20 pts.) For the vector space $\mathcal{P}_3$ of polynomials of degree less than or equal to 3, let $T : \mathcal{P}_3 \to \mathbf{R}$ be the function

$$T(p) = p(2) + p(3) \ .$$

Show that $T$ is a *linear transformation*, and find numbers $a, b$, and $c$ so that

$$T(x + a) = T(x^2 + b) = T(x^3 + c) = 0 \ .$$

We want: $T(p+q) = T(p) + T(q)$, $T(cp) = cT(p)$ for $c \in \mathbb{R}$, $p, q \in \mathcal{P}_3$.

But

$$T(p+q) = (p+q)(2) + (p+q)(3)$$
$$= (p(2) + q(2)) + (p(3) + q(3)) = (p(2) + p(3)) + (q(2) + q(3))$$
$$= T(p) + T(q) \ \checkmark$$

$$T(cp) = (cp)(2) + (cp)(3) = c(p(2)) + c(p(3))$$
$$= c(p(2) + p(3)) = cT(p) \ \checkmark$$

So: $T$ is a linear transformation.

$$T(x+a) = (2+a) + (3+a) = 2a + 5 = 0 \text{ for } a = \frac{-5}{2}$$
$$T(x^2+b) = (4+b) + (9+b) = 2b + 13 = 0 \text{ for } b = -13/2$$
$$T(x^3+c) = (8+c) + (27+c) = 2c + 35 = 0 \text{ for } c = -35/2$$

So $T(x - \frac{5}{2}) = T(x^2 - \frac{13}{2}) = T(x^3 - \frac{35}{2}) = 0$. ///

**4.** (20 pts.) Show that the collection of vectors $W = \{(a\ b\ c)^T \in \mathbf{R}^3\ :\ 3a - 2b + c = 0\}$ is a *subspace* of $\mathbf{R}^3$, and find a *basis* for $W$.

Need: $\vec{v}, \vec{w} \in W \Rightarrow \vec{v}+\vec{w} \in W$

$\vec{v} \in W,\ c \in \mathbb{R} \Rightarrow c\vec{v} \in W$

$\vec{v} = \begin{pmatrix} a \\ b \\ c \end{pmatrix},\ \vec{w} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$, so $3a-2b+c=0$, $3x-2y+z=0$, then

$\vec{v}+\vec{w} = \begin{pmatrix} a+x \\ b+y \\ c+z \end{pmatrix}$, and $3(a+x) - 2(b+y) + (c+z)$
$= (3a-2b+c) + (3x-2y+z) = 0+0 = 0$

so $\vec{v}+\vec{w} \in W$. ✓

$k\vec{v} = \begin{pmatrix} ka \\ kb \\ kc \end{pmatrix}$, and $3(ka) - 2(kb) + (kc)$
$= k(3a-2b+c) = k(0) = 0$

so $k\vec{v} \in W$ ✓ So $W$ is a subspace.

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W looks like a nullspace!

$(3\ {-2}\ 1)\begin{pmatrix} a \\ b \\ c \end{pmatrix} = 0$, so $W = \mathrm{Nul}\,(3\ {-2}\ 1)$.

Basis: row reduce! $(3\ \ {-2}\ \ 1) \longrightarrow (1\ \ {-\tfrac{2}{3}}\ \ \tfrac{1}{3})$

$x - \tfrac{2}{3}y + \tfrac{1}{3}z = 0$

Basis: $\begin{pmatrix} 2/3 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -1/3 \\ 0 \\ 1 \end{pmatrix}$

$x = \tfrac{2}{3}y - \tfrac{1}{3}z$

$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} \tfrac{2}{3}y - \tfrac{1}{3}z \\ y \\ z \end{pmatrix} = y\begin{pmatrix} 2/3 \\ 1 \\ 0 \end{pmatrix} + z\begin{pmatrix} -1/3 \\ 0 \\ 1 \end{pmatrix}$

**5.** (15 pts.) If a $5 \times 8$ matrix $C$ has rank equal to 4, what is the dimension of its nullspace (and why does it have that value?)?

$4 = \text{rank}(C) = \dim(\text{Col}(C)) =$ # of pivots in (R)REF of $C$. $C$ has 8 columns, so with 4 pivots, this means it has 4 free variables in (R)REF.

But $\dim(\text{Nul}(C)) =$ # of free variables in (R)REF,

so $\dim(\text{Nul}(C)) = \boxed{4} = 8 - 4$.

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**3.** (25 pts.) Find bases for the column, row, and nullspaces of the matrix

$$B = \begin{pmatrix} 1 & 2 & 1 & -1 \\ 3 & -1 & 2 & 3 \\ -3 & 8 & -1 & -9 \\ 5 & 3 & 4 & 1 \end{pmatrix}.$$

Row reduce!

$$\begin{pmatrix} 1 & 2 & 1 & -1 \\ 3 & -1 & 2 & 3 \\ -3 & 8 & -1 & -9 \\ 5 & 3 & 4 & 1 \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & 2 & 1 & -1 \\ 0 & -7 & -1 & 6 \\ 0 & 14 & 2 & -12 \\ 0 & -7 & -1 & 6 \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & 2 & 1 & -1 \\ 0 & -7 & -1 & 6 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\longrightarrow \begin{pmatrix} 1 & 2 & 1 & -1 \\ 0 & 1 & 1/7 & -6/7 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \longrightarrow \begin{array}{c} \begin{pmatrix} 1 & 0 & 5/7 & 5/7 \\ 0 & 1 & 1/7 & -6/7 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \\ \uparrow\ \uparrow \quad \uparrow\ \uparrow \\ \text{pivots} \quad \text{free.} \end{array}$$

$$x + 5/7 z + 5/7 w = 0$$
$$y + 1/7 z - 6/7 w = 0$$
$$x = -5/7 z - 5/7 w$$
$$y = -1/7 z + 6/7 w$$

$$\begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix} = \begin{pmatrix} -5/7 z - 5/7 w \\ -1/7 z + 6/7 w \\ z \\ w \end{pmatrix}$$

So: $\begin{pmatrix} 1 \\ 3 \\ -3 \\ 5 \end{pmatrix}, \begin{pmatrix} 2 \\ -1 \\ 8 \\ 3 \end{pmatrix}$ = basis for Col(B)

$\begin{pmatrix} 1 \\ 0 \\ 5/7 \\ 5/7 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 1/7 \\ -6/7 \end{pmatrix}$ = basis for Row(B)

$\begin{pmatrix} -5/7 \\ -1/7 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -5/7 \\ 6/7 \\ 0 \\ 1 \end{pmatrix}$ = basis for Nul(B)

# Math 314 Matrix Theory

## Exam 2 Solutions

Show all work. How you get your answer is just as important, if not more important, than the answer itself. If you think it, write it!

**1.** (20 pts.) For which value(s) of $x$ is the matrix $A = \begin{pmatrix} 1 & 2 & 1 \\ 2 & x & 2 \\ 1 & 1 & x \end{pmatrix}$ not invertible?

We can row reduce the matrix, and look for a column without a pivot (for some values of $x$):

$$A = \begin{pmatrix} 1 & 2 & 1 \\ 2 & x & 2 \\ 1 & 1 & x \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & 2 & 1 \\ 0 & x-4 & 0 \\ 0 & -1 & x-1 \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & 2 & 1 \\ 0 & -1 & x-1 \\ 0 & x-4 & 0 \end{pmatrix}$$

$$\longrightarrow \begin{pmatrix} 1 & 2 & 1 \\ 0 & 1 & 1-x \\ 0 & x-4 & 0 \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & 2 & 1 \\ 0 & 1 & 1-x \\ 0 & 0 & (1-x)(4-x) \end{pmatrix}$$

This has three pivots, unless $(1-x)(4-x)=0$, that is, unless $x=1$ or $x=4$. So for these values of $x$ the matrix will not be invertible; for any other value of $x$ is will be invertible.

Or: we can use the fact that $A$ is invertible $\Leftrightarrow \det(A) \neq 0$. So we compute:

$$\det(A) = (1)\det\begin{pmatrix} x & 2 \\ 1 & x \end{pmatrix} - (2)\det\begin{pmatrix} 2 & 2 \\ 1 & x \end{pmatrix} + (1)\det\begin{pmatrix} 2 & x \\ 1 & 1 \end{pmatrix} =$$

$$(1)(x^2-2) - (2)(2x-2) + (1)(2-x) = x^2 - 2 - 4x + 4 + 2 - x = x^2 - 5x + 4,$$

so $A$ is not invertible precisely when $x^2-5x+4=0$.

But since $x^2-5x+4 = (x-1)(x-4) = 0$ for $x=1$ and $x=4$, we have $A$ is not invertible precisely when $x=1$ or $x=4$.

**2.** (20 pts.) Does the collection of vectors

$$W = \left\{ \begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix} : x - 2y + 4z + 3w = 0 \right\}$$

form a vector space (using the usual addition and scalar multiplication of vectors)? Explain why or why not.

We can approach this two ways: the short way is to note that $W$ is the nullspace of the matrix $\begin{pmatrix} 1 & -2 & 4 & 3 \end{pmatrix}$, and a nullspace is a subspace (of $\mathbf{R}^4$), and so is a vector space. Or we do it the longer way:

If $\begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix}$ and $\begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix}$ are both in $W$, then $x-2y+4z+3w = 0$ and $a-2b+4c+3d = 0$, so $(x+a)-2(y+b)+4(z+c)+3(w+d) = (x-2y+4z+3w)+(a-2b+4c+3d) = 0+0 = 0$,

so $\begin{pmatrix} x+a \\ y+b \\ z+c \\ w+d \end{pmatrix} = \begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix} + \begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix}$ is in $W$, and so $W$ is closed under vector addition.

Similarly, if $\begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix}$ is in $W$ and $c \in \mathbf{R}$, then

$(cx) - 2(cy) + 4(cz) + 3(cw) = c(x - 2y + 4z + 3w) = c(0) = 0$, so $\begin{pmatrix} cx \\ cy \\ cz \\ cw \end{pmatrix} = c \begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix}$ is in $W$, and so $W$ is closed under scalar multiplication.

Since $W$ is closed under both vector addition and scalar multiplication, $W$ is a subspace of $\mathbf{R}^4$, and so is a vector space.

**3.** (25 pts.) Use a superaugmented matrix to express the column space of the matrix

$$A = \begin{pmatrix} 1 & 2 & 4 \\ 2 & 1 & 5 \\ 2 & 2 & 6 \end{pmatrix}$$

as the nullspace of another matrix $B$, and use this to decide if the systems of equations $A\vec{x} = \vec{b}$ are consistent, for the vectors $\vec{b}$ equal to

$$\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}, \begin{pmatrix} 1 \\ 5 \\ 4 \end{pmatrix}, \text{ and } \begin{pmatrix} 3 \\ 2 \\ 1 \end{pmatrix}.$$

We start by row reducing $(A|I_3)$:

$$A = \begin{pmatrix} 1 & 2 & 4 & | & 1 & 0 & 0 \\ 2 & 1 & 5 & | & 0 & 1 & 0 \\ 2 & 2 & 6 & | & 0 & 0 & 1 \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & 2 & 4 & | & 1 & 0 & 0 \\ 0 & -3 & -3 & | & -2 & 1 & 0 \\ 0 & -2 & -2 & | & -2 & 0 & 1 \end{pmatrix}$$

$$\longrightarrow \begin{pmatrix} 1 & 2 & 4 & | & 1 & 0 & 0 \\ 0 & 1 & 1 & | & 2/3 & -1/3 & 0 \\ 0 & -2 & -2 & | & -2 & 0 & 1 \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & 2 & 4 & | & 1 & 0 & 0 \\ 0 & 1 & 1 & | & 2/3 & -1/3 & 0 \\ 0 & 0 & 0 & | & -2/3 & -2/3 & 1 \end{pmatrix}$$

$$\longrightarrow \begin{pmatrix} 1 & 2 & 4 & | & 1 & 0 & 0 \\ 0 & 1 & 1 & | & 2/3 & -1/3 & 0 \\ 0 & 0 & 0 & | & -2 & -2 & 3 \end{pmatrix}$$

This tells us that $\begin{pmatrix} a \\ b \\ c \end{pmatrix}$ will give a consistent system of equations precisely when

$$\begin{pmatrix} -2 & -2 & 3 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \end{pmatrix} = (-2a - 2b + 3c) = (0),$$

i.e., $\begin{pmatrix} a \\ b \\ c \end{pmatrix}$ is in the nullspace of $B = \begin{pmatrix} -2 & -2 & 3 \end{pmatrix}$.

So the column space of $A$ is the nullspace of $B = \begin{pmatrix} -2 & -2 & 3 \end{pmatrix}$.

Using this, we can test the three vectors we were given:

$\begin{pmatrix} -2 & -2 & 3 \end{pmatrix} \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} = -2 - 4 + 9 = -6 + 9 = 3 \neq 0,$
so this does not give a consistent system of equations.

$\begin{pmatrix} -2 & -2 & 3 \end{pmatrix} \begin{pmatrix} 1 \\ 5 \\ 4 \end{pmatrix} = -2 - 10 + 12 = -12 + 12 = 0,$
so this gives a consistent system of equations.

$\begin{pmatrix} -2 & -2 & 3 \end{pmatrix} \begin{pmatrix} 3 \\ 2 \\ 1 \end{pmatrix} = -6 - 4 + 3 = -10 + 3 = -7 \neq 0,$
so this does not give a consistent system of equations.

**4.** (25 pts.) Find a collection from among the vectors

$$\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}, \begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 2 \\ 2 \\ 3 \end{pmatrix}, \begin{pmatrix} 3 \\ 1 \\ 2 \end{pmatrix}$$

that forms a basis for $\mathbf{R}^3$, and express the <u>remaining</u> vectors as linear combinations of your chosen basis vectors.

[Hint: your work for the first part should tell you how to answer the second part!]

To find a basis, we need linear independence and spanning $\mathbf{R}^3$, so we put the vectors together in a matrix and row reduce!

$$A = \begin{pmatrix} 1 & 2 & 2 & 3 \\ 2 & 1 & 2 & 1 \\ 3 & 1 & 3 & 2 \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & 2 & 2 & 3 \\ 0 & -3 & -2 & -5 \\ 0 & -5 & -3 & -7 \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & 2 & 2 & 3 \\ 0 & 1 & 2/3 & 5/3 \\ 0 & -5 & -3 & -7 \end{pmatrix}$$

$$\longrightarrow \begin{pmatrix} 1 & 2 & 2 & 3 \\ 0 & 1 & 2/3 & 5/3 \\ 0 & 0 & 1/3 & 4/3 \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & 2 & 2 & 3 \\ 0 & 1 & 2/3 & 5/3 \\ 0 & 0 & 1 & 4 \end{pmatrix}$$

So in row echelon form, there are pivots in the first three columns, so those three columns of $A$ are linearly independent. They also span (a pivot in every row, or 3 linearly independent vectors in $\mathbf{R}^3$ !), so the first three columns form a basis for $\mathbf{R}^3$. To write the remaining vector as a linear combination, we can note that we have done most of the work of solving the needed linear system; just insert a vertical bar before the fourth column!

$$\left(\begin{array}{ccc|c} 1 & 2 & 2 & 3 \\ 2 & 1 & 2 & 1 \\ 3 & 1 & 3 & 2 \end{array}\right) \longrightarrow \left(\begin{array}{ccc|c} 1 & 2 & 2 & 3 \\ 0 & 1 & 2/3 & 5/3 \\ 0 & 0 & 1 & 4 \end{array}\right) \longrightarrow \left(\begin{array}{ccc|c} 1 & 2 & 2 & 3 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 4 \end{array}\right)$$

$$\longrightarrow \begin{pmatrix} 1 & 2 & 0 & | & -5 \\ 0 & 1 & 0 & | & -1 \\ 0 & 0 & 1 & | & 4 \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & 0 & 0 & | & -3 \\ 0 & 1 & 0 & | & -1 \\ 0 & 0 & 1 & | & 4 \end{pmatrix}$$

which tells us that $\begin{pmatrix} 3 \\ 1 \\ 2 \end{pmatrix} = (-3)\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} + (-1)\begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix} + (4)\begin{pmatrix} 2 \\ 2 \\ 3 \end{pmatrix}$.

Another approach, which many of you followed, is to find a basis for the nullspace of this matrix $A$. From the RREF, we have $x_1 - 3x_4 = 0$, $x_2 - x_4 = 0$, and $x_3 + 4x_4 = 0$, so

$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 3x_4 \\ x_4 \\ -4x_4 \\ x_4 \end{pmatrix} = x_4 \begin{pmatrix} 3 \\ 1 \\ -4 \\ 1 \end{pmatrix}$, so $\begin{pmatrix} 3 \\ 1 \\ -4 \\ 1 \end{pmatrix}$ is a basis for the nullspace of $A$. But this means that

$$(3)\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} + (1)\begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix} + (-4)\begin{pmatrix} 2 \\ 2 \\ 3 \end{pmatrix} + (1)\begin{pmatrix} 3 \\ 1 \\ 2 \end{pmatrix} = \vec{0},$$

which means that $\begin{pmatrix} 3 \\ 1 \\ 2 \end{pmatrix} = (-3)\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} + (-1)\begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix} + (4)\begin{pmatrix} 2 \\ 2 \\ 3 \end{pmatrix}$.

**5.** (10 pts.) Show why if $A$ and $B$ are matrices so that $AB$ makes sense, and the matrix $AB$ has <u>linearly</u> <u>independent</u> columns, then $B$ must have linearly independent columns.

[Hint: What does the conclusion, about $B$, say about systems of linear equations?]

We want to say that $B$ has linearly independent columns, which means that the only linear combination of the columns that equals the vector $\vec{0}$ is the all-0 linear combination. In matrix terms, if $B\vec{x} = \vec{0}$, then we must have $\vec{x} = \vec{0}$.

But if we then suppose that $B\vec{x} = \vec{0}$, then $A(B\vec{x}) = A\vec{0} = \vec{0}$. But $A(B\vec{x}) = (AB)\vec{x}$, and so we know that $(AB)\vec{x} = \vec{0}$. But the columns of $AB$ are linearly independent!, and so the same line of reasoning shows that we must have $\vec{x} = \vec{0}$.

So, if $B\vec{x} = \vec{0}$ then $((AB)\vec{x} = \vec{0}$, and so) $\vec{x} = \vec{0}$, showing that the columns ob $B$ are linearly independent.

An alternate approach, taken by some, is to think of matrix multplication as a linear transformation $T_A$, etc., and note that $T_{AB} = T_A \circ T_B$. Having linearly independent columns amounts (by essentially the same reasoning as above) to saying that $T_{AB}$ is a one-to-one function [$T_{AB}(\vec{x}) = T_{AB}(\vec{y})$ means $T_{AB}(\vec{x} - \vec{y}) = \vec{0}$, so $\vec{x} - \vec{y} = \vec{0}$, so $\vec{x} = \vec{y}$.] But if $T_B$ is <u>not</u> one-to-one, then $T_A \circ T_B$ cannot be; $T_B(\vec{x}) = T_B(\vec{y})$ means that $T_{AB}(\vec{x}) = T_A(T_B(\vec{x})) = T_A(T_B(\vec{y})) = T_{AB}(\vec{y})$. So $T_{AB}$ one-to-one implies that $T_B$ must be one-t-one, so $B$ has linearly independent columns.