Math 314/814 Matrix Theory

Final practice problems

1. Find bases for the column space and row space of the matrix

$$A = \begin{pmatrix} 1 & 1 & 2 & 1 \\ 2 & 0 & 6 & 2 \\ 2 & -2 & 12 & -14 \\ -1 & -2 & 1 & -9 \end{pmatrix}$$

2. Find a basis for $\mathbb{R}^3$ which includes, among its vectors, a basis for the nullspace of the matrix

$$A = \begin{pmatrix} 3 & 1 & 3 \\ 2 & 2 & 2 \end{pmatrix}$$

3. Find the inverse of the matrix $A = \begin{pmatrix} 1 & 3 & 2 \\ 3 & 2 & 1 \\ 2 & 3 & 1 \end{pmatrix}$.

4. For the matrix $A = \begin{bmatrix} 9 & -4 \\ 20 & -9 \end{bmatrix}$, what is A^{2008} ?

(Hint: knowing its eigenvalues might help...)

5. The vectors $\vec{v}_1 = \begin{bmatrix} 1 \\ 0 \\ 1 \\ 1 \end{bmatrix}$, $\vec{v}_2 = \begin{bmatrix} 2 \\ 0 \\ -1 \\ 0 \end{bmatrix}$, $\vec{v}_3 = \begin{bmatrix} -1 \\ 0 \\ 1 \\ 0 \end{bmatrix}$ are linearly independent (you need not verify this).

Find the vector in $W = \text{span}(\vec{v}_1, \vec{v}_2, \vec{v}_3)$ which is closest to the vector $\vec{b} = \begin{bmatrix} 3 \\ 4 \\ 5 \\ 6 \end{bmatrix}$.

6. Find the line $y = ax + b$ that best approximates the data points

$$\{(-2,3),(0,5),(1,7)\}.$$

7. Show that if A, B are a pair of $m \times n$ matrices, then the collection of vectors

$$W = \{\vec{v} \in \mathbb{R}^n : A\vec{v} = B\vec{v}\}$$

is a subspace of $\mathbb{R}^m$.

8. Use Gram-Schmidt orthogonalization, starting with the basis

$$\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$$

for $\mathbb{R}^3$, to build an orthogonal basis for $\mathbb{R}^3$.

9. For what values of x is the matrix $A = \begin{pmatrix} x & 1 & 3 \\ 3 & 1 & x \\ 0 & -1 & x \end{pmatrix}$ invertible?

10. Find the eigenvalues of the matrix

$$A = \begin{pmatrix} 2 & 3 \\ 4 & 3 \end{pmatrix}$$

and, for each eigenvalue, find a basis for its eigenspace.

11. Explain why the function $f : \mathbb{R}^2 \to \mathbb{R}^2$ given by

$$f(x, y) = (x - y, x^2 + y^2)$$

is **not** a linear transformation.

12. Find the matrix A so that $T = T_A$, where $T : \mathbb{R}^2 \to \mathbb{R}^2$ is the linear transformation which given a vector $[x \;\; y]^T$ returns the vector $[y \;\; x]^T$. Geometrically, what does this transformation do?

13. Find bases for the column space of the matrix $A = \begin{pmatrix} 2 & 3 & 4 \\ 5 & 7 & 9 \\ 1 & 4 & 7 \end{pmatrix}$, by

(a) row reducing the matrix A ,

(b) row reducing the transpose A^T of the matrix A .

14. Find the value of Ax closest to b, where

$$A = \begin{pmatrix} 1 & 2 \\ 1 & 0 \\ 0 & 1 \end{pmatrix} \qquad \text{and} \qquad b = \begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix}$$

15. Find the determinant of the matrix

$$A = \begin{pmatrix} 1 & 2 & -7 \\ 1 & 0 & -3 \\ 3 & 1 & 3 \end{pmatrix}$$

Based on this, find the determinants of the matrices $B = A^{-1}$, $C = A^T$, and $D = A^T A$. (Hint: you don't need to compute these matrices....)

16. Find the orthogonal complement of the subspace W of R^4 spanned by the vectors

$$\begin{pmatrix} 1 \\ -1 \\ 0 \\ 2 \end{pmatrix} \qquad \text{and} \qquad \begin{pmatrix} -1 \\ 1 \\ 1 \\ 2 \end{pmatrix}$$

17. Find bases for the column space, row space, and nullspace of the matrix

$$A = \begin{pmatrix} 1 & -1 & 2 \\ 2 & 1 & 2 \\ 1 & 5 & -2 \end{pmatrix} .$$

18. Find the inverse of the matrix $A = \begin{pmatrix} 1 & -1 & 3 \\ 1 & 2 & 2 \\ 1 & 2 & 1 \end{pmatrix}$.

19. For which value(s) of x do the vectors

$$\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} , \begin{pmatrix} -1 \\ x \\ 2 \end{pmatrix} , \begin{pmatrix} x \\ 2 \\ 2 \end{pmatrix} \qquad \textbf{span } \mathbb{R}^3 \ ?$$

20. Use the Gram-Schmidt orthogonalization process to construct an orthogonal set from the (you may assume linearly independent) vectors

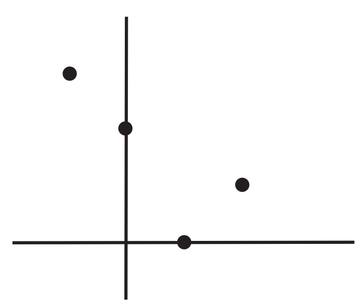
$$\vec{w}_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} , \vec{w}_2 = \begin{pmatrix} 1 \\ 1 \\ -1 \\ 1 \end{pmatrix} , \vec{w}_3 = \begin{pmatrix} 1 \\ 1 \\ -1 \\ -1 \end{pmatrix} .$$

21. Find the orthogonal projection of the vector $\vec{v} = \begin{pmatrix} 3 \\ 2 \\ 1 \end{pmatrix}$ onto the column space of the matrix

$$A = \begin{pmatrix} 1 & -1 \\ 1 & 1 \\ 2 & 1 \end{pmatrix} .$$

22. Find the line $y = ax + b$ that gives the least squares best fit to the data points

$$(-1, 3) \ , \ (0, 2) \ , \ (1, 0) \ , \ (2, 1) \ .$$



23. Find the eigenvalues, and associated eigenbases, for the matrix

$$A = \begin{pmatrix} 0 & 2 \\ 3 & 1 \end{pmatrix} .$$