Name:

Math 314/814 Matrix Theory
Final Exam

Show all work. Include all steps necessary to arrive at an answer unaided by a mechanical computational device. The steps you take to your answer are just as important, if not more important, than the answer itself. If you think it, write it!

1. (20 pts.) Find bases for the column space and row space of the matrix

$$A = \begin{pmatrix} 1 & 1 & 2 & 1 \\ 2 & 0 & 6 & 2 \\ 2 & -2 & 12 & -14 \\ -1 & -2 & 1 & -9 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 1 & 2 & 1 \\ 2 & 0 & 6 & 2 \\ 2 & -2 & 12 & -14 \\ -1 & -2 & 1 & -9 \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & 1 & 2 & 1 \\ 0 & -2 & 2 & 0 \\ 0 & -4 & 8 & -16 \\ 0 & -1 & 3 & -8 \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & 1 & 2 & 1 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & +4 & -16 \\ 0 & 0 & 2 & -8 \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & 1 & 2 & 1 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 1 & -4 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\longrightarrow \begin{pmatrix} 1 & 0 & 3 & 1 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 1 & -4 \\ 0 & 0 & 0 & 0 \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & 0 & 0 & 13 \\ 0 & 1 & 0 & -4 \\ 0 & 0 & 1 & -4 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\begin{array}{c} / \ 1 \ 1 \ \backslash \\ \text{pivots} \end{array}$$

So Basis for col(A): $\begin{pmatrix} 1 \\ 2 \\ 2 \\ -1 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ -2 \\ -2 \end{pmatrix}, \begin{pmatrix} 2 \\ 6 \\ 12 \\ -1 \end{pmatrix}$

Basis for row(A): $\begin{pmatrix} 1 \\ 0 \\ 0 \\ 13 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \\ -4 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \\ -4 \end{pmatrix}$

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2. (25 pts.) Find a basis for $\mathbb{R}^3$ which includes, among its vectors, a basis for the nullspace of

$$A = \begin{pmatrix} 3 & 1 & 3 \\ 2 & 2 & 2 \end{pmatrix}$$

$$\begin{pmatrix} 3 & 1 & 3 \\ 2 & 2 & 2 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 1 & 1 \\ 3 & 1 & 3 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 1 & 1 \\ 0 & -2 & 0 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$$

so $x + z = 0$ $x = -z$ basis for null(A) : $\begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$ $\underset{free}{\downarrow}$

$y = 0$

$$\begin{pmatrix} -1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

$\underset{pivots}{\uparrow\uparrow\uparrow}$

so $\begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$ are a basis for $\mathbb{R}^3$

$\underset{basis\ for\ null(A).}{\uparrow}$

3. (20 pts.) Find the inverse of the matrix $A = \begin{pmatrix} 1 & 3 & 2 \\ 3 & 2 & 1 \\ 2 & 3 & 1 \end{pmatrix}$.

$$\left(\begin{array}{ccc|ccc} 1 & 3 & 2 & 1 & 0 & 0 \\ 3 & 2 & 1 & 0 & 1 & 0 \\ 2 & 3 & 7 & 0 & 0 & 1 \end{array}\right) \longrightarrow \left(\begin{array}{ccc|ccc} 1 & 3 & 2 & 1 & 0 & 0 \\ 0 & -7 & -5 & -3 & 1 & 0 \\ 0 & -3 & -3 & -2 & 0 & 1 \end{array}\right)$$

$$\longrightarrow \left(\begin{array}{ccc|ccc} 1 & 3 & 2 & 1 & 0 & 0 \\ 0 & 1 & 1 & +2/3 & 0 & -1/3 \\ 0 & -7 & -5 & -3 & 1 & 0 \end{array}\right) \longrightarrow \left(\begin{array}{ccc|ccc} 1 & 3 & 2 & 1 & 0 & 0 \\ 0 & 1 & 1 & +2/3 & 0 & -1/3 \\ 0 & 0 & 2 & & & \end{array}\right)$$

$$5/3 \quad 1 \quad -7/3$$

$$\longrightarrow \left(\begin{array}{ccc|ccc} 1 & 3 & 2 & & & \\ 0 & 1 & 1 & & & \\ 0 & 0 & 2 & & & \end{array}\right)$$

$$5/3 \quad 1 \quad -7/3$$

$$\longrightarrow \left(\begin{array}{ccc|ccc} 1 & 3 & 2 & 1 & 0 & 0 \\ 0 & 1 & 0 & -1/6 & -1/2 & 5/6 \\ 0 & 0 & 1 & 5/6 & 1/2 & -7/6 \end{array}\right) \longrightarrow \left(\begin{array}{ccc|ccc} 1 & 3 & 0 & -2/3 & -1 & +7/3 \\ 0 & 1 & 0 & -1/6 & -1/2 & 5/6 \\ 0 & 0 & 1 & 5/6 & 1/2 & -7/6 \end{array}\right)$$

$$\longrightarrow \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & -1/6 & 1/2 & -1/6 \\ 0 & 1 & 0 & -1/6 & -1/2 & 5/6 \\ 0 & 0 & 1 & 5/6 & 1/2 & -7/6 \end{array}\right)$$

$$\text{So} \quad A^{-1} = \begin{pmatrix} -1/6 & 1/2 & -1/6 \\ -1/6 & -1/2 & 5/6 \\ 5/6 & 1/2 & -7/6 \end{pmatrix}$$

4. (25 pts.) For the matrix $A = \begin{bmatrix} 9 & -4 \\ 20 & -9 \end{bmatrix}$, what is A^{2008} ?

(Hint: knowing its eigenvalues might help...)

$$\det(A - \lambda I) = \begin{vmatrix} 9-\lambda & -4 \\ 20 & -9-\lambda \end{vmatrix} = (9-\lambda)(-9-\lambda) - (-4)(20)$$

$$= \lambda^2 - 9\lambda + 9\lambda - 81 + 80 = \lambda^2 - 1$$

$$= (\lambda-1)(\lambda+1)$$

So e-values are $\lambda = 1$, $\lambda = -1$ A is diagonalizable!

$\Big[$ Stop! So $AP = PD = P \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ for some invertible P

$A = P \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} P^{-1}$ So $A^{2008} = P \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}^{2008} P^{-1} = P \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} P^{-1}$

$= PP^{-1} = I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$. $\Big]$

$\lambda = 1$: $\begin{bmatrix} 8 & -4 \\ 20 & -10 \end{bmatrix} \rightarrow \begin{bmatrix} 2 & -1 \\ 20 & -10 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & -\frac{1}{2} \\ 0 & 0 \end{bmatrix}$

$x = \frac{1}{2}y$ $2x = y$ $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$ e-basis.

$\lambda = -1$: $\begin{bmatrix} 10 & -4 \\ 20 & -8 \end{bmatrix} \rightarrow \begin{bmatrix} 5 & -2 \\ 0 & 0 \end{bmatrix}$ $5x = 2y$ $\begin{bmatrix} 2 \\ 5 \end{bmatrix}$

So $AP = PD = P \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$ & $P = \begin{bmatrix} 1 & 2 \\ 2 & 5 \end{bmatrix}$. So $\det P = 5 - 4 = 1$

$A = PDP^{-1}$, so $A^{2008} =$ $P^{-1} = \begin{bmatrix} 5 & -2 \\ -2 & 1 \end{bmatrix}$

$\begin{bmatrix} 1 & 2 \\ 2 & 5 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}^{2008} \begin{bmatrix} 5 & -2 \\ -2 & 1 \end{bmatrix}$

4   $= \cdots = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$.

5. (25 pts.) The vectors $\vec{v}_1 = \begin{bmatrix} 1 \\ 0 \\ 1 \\ 1 \end{bmatrix}$, $\vec{v}_2 = \begin{bmatrix} 2 \\ 0 \\ -1 \\ \cancel{1}\,0 \end{bmatrix}$, $\vec{v}_3 = \begin{bmatrix} -1 \\ 0 \\ 1 \\ 0 \end{bmatrix}$ are linearly independent (you need not verify this).

Find the vector in $W = \text{span}(\vec{v}_1, \vec{v}_2, \vec{v}_3)\}$ which is closest to the vector $\vec{b} = \begin{bmatrix} 3 \\ 4 \\ 5 \\ 6 \end{bmatrix}$.

$$W = \text{col}(A) \qquad A = \begin{pmatrix} 1 & 2 & -1 \\ 0 & 0 & 0 \\ 1 & -1 & 1 \\ 1 & 0 & 0 \end{pmatrix}$$

$$\text{closest} = \vec{v} = A\vec{x} \quad \Longleftrightarrow \quad A^TA\vec{x} = A^T\vec{b}$$

$$A^TA = \begin{pmatrix} 1 & 0 & 1 & 1 \\ 2 & 0 & -1 & 0 \\ -1 & 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 2 & -1 \\ 0 & 0 & 0 \\ 1 & -1 & 1 \\ 1 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 3 & 1 & 0 \\ 1 & 5 & -3 \\ 0 & -3 & 2 \end{pmatrix}$$

$$A^T\vec{b} = \begin{pmatrix} 1 & 0 & 1 & 1 \\ 2 & 0 & -1 & 0 \\ -1 & 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} 3 \\ 4 \\ 5 \\ 6 \end{pmatrix} = \begin{pmatrix} 14 \\ 1 \\ 2 \end{pmatrix}$$

$$\left(\begin{array}{ccc|c} 3 & 1 & 0 & 14 \\ 1 & 5 & -3 & 1 \\ 0 & -3 & 2 & 2 \end{array} \right) \rightarrow \left(\begin{array}{ccc|c} 1 & 5 & -3 & 1 \\ 0 & -14 & 9 & 11 \\ 0 & -3 & 2 & 2 \end{array} \right) \rightarrow \left(\begin{array}{ccc|c} 1 & 5 & -3 & 1 \\ 0 & 1 & -2/3 & -2/3 \\ 0 & -14 & 9 & 11 \end{array} \right)$$

$$\rightarrow \left(\begin{array}{ccc|c} 1 & 5 & -3 & 1 \\ 0 & 1 & -2/3 & -2/3 \\ 0 & 0 & -1/3 & 5/3 \end{array} \right) \rightarrow \left(\begin{array}{ccc|c} 1 & 5 & -3 & 1 \\ 0 & 1 & -2/3 & -2/3 \\ 0 & 0 & 1 & -5 \end{array} \right) \rightarrow \left(\begin{array}{ccc|c} 1 & 5 & -3 & -14 \\ 0 & 1 & 0 & -4 \\ 0 & 0 & 1 & -5 \end{array} \right)$$

$$\rightarrow \left(\begin{array}{ccc|c} 1 & 0 & 0 & 6 \\ 0 & 1 & 0 & -4 \\ 0 & 0 & 1 & -5 \end{array} \right) \qquad \text{so} \quad \vec{x} = \begin{pmatrix} 6 \\ -4 \\ -5 \end{pmatrix} \quad \text{so}$$

$$\text{closest } \vec{v} = \begin{pmatrix} 1 & 2 & -1 \\ 0 & 0 & 0 \\ 1 & -1 & 1 \\ 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} 6 \\ -4 \\ -5 \end{pmatrix} = \begin{pmatrix} 3 \\ 0 \\ 5 \\ 6 \end{pmatrix}$$

6. (20 pts.) Find the line $y = ax + b$ that best approximates the data points
$$\{(-2,3), (0,5), (1,7)\}.$$

$$\begin{pmatrix} -2 & 1 \\ 0 & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} \text{ closest to } \begin{pmatrix} 3 \\ 5 \\ 7 \end{pmatrix} = \vec{b}$$

$$A = \begin{pmatrix} -2 & 1 \\ 0 & 1 \\ 1 & 1 \end{pmatrix} \qquad A^T A = \begin{pmatrix} -2 & 0 & 1 \\ 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} -2 & 1 \\ 0 & 1 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} 5 & -1 \\ -1 & 3 \end{pmatrix}$$

$$A^T \vec{b} = \begin{pmatrix} -2 & 0 & 1 \\ 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} 3 \\ 5 \\ 7 \end{pmatrix} = \begin{pmatrix} 1 \\ 15 \end{pmatrix}$$

$$\det(A^T A) = 5 \cdot 3 - (-1)(-1) = 15 - 1 = 14$$

$$(A^T A)^{-1} = \frac{1}{14} \begin{pmatrix} 3 & 1 \\ 1 & 5 \end{pmatrix}$$

$$\begin{pmatrix} a \\ b \end{pmatrix} = (A^T A)^{-1}(A^T \vec{b}) = \frac{1}{14} \begin{pmatrix} 3 & 1 \\ 1 & 5 \end{pmatrix} \begin{pmatrix} 1 \\ 15 \end{pmatrix}$$

$$= \frac{1}{14} \begin{pmatrix} 13 \\ \cancel{} \end{pmatrix} = \begin{pmatrix} 13/14 \\ 5/14 \end{pmatrix}$$

$$= \frac{1}{14} \begin{pmatrix} 18 \\ 76 \end{pmatrix} = \begin{pmatrix} 9/7 \\ 38/7 \end{pmatrix}$$

$$y = \frac{9}{7} a + \frac{38}{7} \quad \text{is the best-fitting line.}$$

7. (15 pts.) Show that if A, B are a pair of $m \times n$ matrices, then the collection of vectors
$$W = \{\vec{v} \in \mathbb{R}^n : A\vec{v} = B\vec{v}\}$$
is a subspace of $\mathbb{R}^m$.

$A\vec{v} = B\vec{v} \iff (A-B)\ A\vec{v} - B\vec{v} = 0 = (A-B)\vec{v}$

$\delta \quad W = \text{null} (A-B)$, which is a subspace!

$\partial r'$ $\vec{0} \in W$: $A\vec{0} = \vec{0} = B\vec{0}$. $\checkmark$

if $\vec{v}, \vec{w} \in W$ then $A\vec{v} = B\vec{v}$, $A\vec{w} = B\vec{w}$, so

$A(\vec{v} + \vec{w}) = A\vec{v} + A\vec{w} = B\vec{v} + B\vec{w} = B(\vec{v} + \vec{w})$, so

$\vec{v} + \vec{w} \in W$.

if $\vec{v} \in W$, $c \in \mathbb{R}$, then $A\vec{v} = B\vec{v}$ so

$A(c\vec{v}) = cA\vec{v} = cB\vec{v} = B(c\vec{v})$ so $c\vec{v} \in W$.

so W is closed under addition an scalar multiplication,

so W is a subspace. $/\!/\!/$

Quiz number 11 Solution

Show all work. How you get your answer is just as important, if not more important, than the answer itself. If you think it, write it!

8.

Use Gram-Schmidt orthogonalization, starting with the basis

$$\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$$

for $\mathbb{R}^3$, to build an orthogonal basis for $\mathbb{R}^3$.

Starting with the vectors $\vec{w}_1, \vec{w}_2, \vec{w}_3$ above, we construct:

$$\vec{v}_1 = \vec{w}_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

$$\vec{v}_2 = \vec{w}_2 - \left(\frac{<\vec{w}_2, \vec{v}_1>}{<\vec{v}_1, \vec{v}_1>} \vec{v}_1 \right) = \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix} - \frac{4}{3} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} -1/3 \\ -1/3 \\ 2/3 \end{bmatrix} = \frac{1}{3} \begin{bmatrix} -1 \\ -1 \\ 2 \end{bmatrix}$$

$$\vec{v}_3 = \vec{w}_3 - \left(\frac{<\vec{w}_3, \vec{v}_1>}{<\vec{v}_1, \vec{v}_1>} \vec{v}_1 + \frac{<\vec{w}_3, \vec{v}_2>}{<\vec{v}_2, \vec{v}_2>} \vec{v}_2 \right) = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} - \left(\frac{6}{3} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + \frac{\frac{1}{3}3}{(\frac{1}{3})^2(6)} (\frac{1}{3}) \begin{bmatrix} -1 \\ -1 \\ 2 \end{bmatrix} \right)$$

$$= \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} - \left(2 \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + \frac{1}{2} \begin{bmatrix} -1 \\ -1 \\ 2 \end{bmatrix} \right) = \begin{bmatrix} 1-2+1/2 \\ 2-2+1/2 \\ 3-2-1 \end{bmatrix} = \begin{bmatrix} -1/2 \\ 1/2 \\ 0 \end{bmatrix} = \frac{1}{2} \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}$$

So our orthogonal basis is $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \frac{1}{3} \begin{bmatrix} -1 \\ -1 \\ 2 \end{bmatrix}, \frac{1}{2} \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}$.

Quiz number 8 Solutions

Show all work. How you get your answer is just as important, if not more important, than the answer itself. If you think it, write it!

9. For what values of x is the matrix

$$A = \begin{pmatrix} x & 1 & 3 \\ 3 & 1 & x \\ 0 & -1 & x \end{pmatrix}$$

invertible?

A is invertible precisely when its determinant is non-zero. So we compute

$$\det(A) = x \cdot \det \begin{pmatrix} 1 & x \\ -1 & x \end{pmatrix} - 3 \cdot \det \begin{pmatrix} 1 & 3 \\ -1 & x \end{pmatrix} + 0 \cdot \det \begin{pmatrix} 1 & 3 \\ 1 & x \end{pmatrix}$$
$$= x(x - (-x)) - 3(x - (-3)) + 0(x - 3) = 2x^2 - 3x - 9 \ .$$

To find out where it is non-zero, we find out where it is zero:

$$2x^2 - 3x - 9 = 0 \text{ for } x = \frac{3 \pm \sqrt{9 + 4 \cdot 2 \cdot 9}}{2 \cdot 2} = \frac{3 \pm \sqrt{81}}{4} \ ,$$
so $x = (3 + 9)/4 = 3$ or $x = (3 - 9)/4 = -3/2$

So, for $x \neq -3/2, 3$, A is invertible.

Quiz number 7 solutions

Show all work. How you get your answer is just as important, if not more important, than the answer itself. If you think it, write it!

10. Find the eigenvalues of the matrix

$$A = \begin{pmatrix} 2 & 3 \\ 4 & 3 \end{pmatrix}$$

and, for each eigenvalue, find a basis for its eigenspace.

To find the eigenvalues, we solve the equation

$$(2 - \lambda)(3 - \lambda) - (3)(4) = 0 = \lambda^2 - 5\lambda + 6 - 12 = \lambda^2 - 5\lambda - 6 = (\lambda - 6)(\lambda + 1),$$

so $\lambda = 6$ or $\lambda = -1$. These are our eigenvalues.

To find bases for the eigenspaces we find bases for the nullspaces of $A - 6I$ and $A - (-1)I = A + I$, by row reducing:

$$A - 6I = \begin{pmatrix} 2 - 6 & 3 \\ 4 & 3 - 6 \end{pmatrix} = \begin{pmatrix} -4 & 3 \\ 4 & -3 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & -3/4 \\ 4 & -3 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & -3/4 \\ 0 & 0 \end{pmatrix}$$

so the second variable is free and we have $x - (3/4)y = 0$ so $x = (3/4)y$.

So E_6 has basis $\begin{bmatrix} 3/4 \\ 1 \end{bmatrix}$ (or any non-zero multiple of this).

$$A + I = \begin{pmatrix} 2 + 1 & 3 \\ 4 & +1 \end{pmatrix} = \begin{pmatrix} 3 & 3 \\ 4 & 4 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 1 \\ 4 & 4 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}$$

so the second variable is again free and we have $x + y = 0$ so $x = -y$.

So E_{-1} has basis $\begin{bmatrix} -1 \\ 1 \end{bmatrix}$ (or any non-zero multiple of this).

We can check these answers:

$$A \begin{bmatrix} 3/4 \\ 1 \end{bmatrix} = \begin{bmatrix} 2(3/4) + 3(1) \\ 4(3/4) + 3(1) \end{bmatrix} = \begin{bmatrix} 9/2 \\ 6 \end{bmatrix} = 6 \begin{bmatrix} 3/4 \\ 1 \end{bmatrix}, \text{ and}$$

$$A \begin{bmatrix} -1 \\ 1 \end{bmatrix} = \begin{bmatrix} 2(-1) + 3(1) \\ 4(-1) + 3(1) \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \end{bmatrix} = (-1) \begin{bmatrix} -1 \\ 1 \end{bmatrix}, \text{ as desired.}$$

Quiz number 6 Solutions

Show all work. How you get your answer is just as important, if not more important, than the answer itself. If you think it, write it!

II.

1. Explain why the function $f : \mathbf{R}^2 \to \mathbf{R}^2$ given by
$$f(x, y) = (x - y, x^2 + y^2)$$
is **not** a linear transformation.

$f((x, y) + (a, b)) = f(x + a, y + b) = ((x + a) - (y + b), (x + a)^2 + (y + b)^2)$
$= ((x - y) + (a - b), (x^2 + y^2) + (a^2 + b^2) + (2ax + 2yb))$
$= (x - y, x^2 + y^2) + (a - b, a^2 + b^2) + (0, 2ax + 2yb) = f(x, y) + f(a, b) + (0, 2ax + 2yb)$,
which does **not** equal $f(x, y) + f(a, b)$ in general, e.g. for $x = y = a = b = 1$.

Alternatively, $f(c(x, y)) = f(cx, cy) = (cx - cy, (cx)^2 + (cy)^2) = (cx - cy, c^2 x^2 + c^2 y^2)$,
while
$cf(x, y) = c(x - y, x^2 + y^2) = (cx - cy, cx^2 + cy^2)$, which does **not** equal $f(c(x, y))$ in general, since
$c^2 x^2 + c^2 y^2 \neq cx^2 + cy^2$ in general, e.g. for $x = y = 1$ and $c = 672$.

I2.

2. Find the matrix A so that $T = T_A$, where $T : \mathbf{R}^2 \to \mathbf{R}^2$ is the linear transformation which given a vector $[x \; y]^T$ returns the vector $[y \; x]^T$. Geometrically, what does this transformation do?

$$T \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} y \\ x \end{bmatrix} = \begin{bmatrix} 0 \cdot x + 1 \cdot y \\ 1 \cdot x + 0 \cdot y \end{bmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

So $A = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$.

Geometrically, the transformation sends $(0, 1)$ to $(1, 0)$ and sends $(0, 1)$ to $(1, 0)$; in general, the transformation swaps the x- and y-coordinates, which can be accomplished by reflecting the vector across the line $y = x$. So T is reflection across the line $y = x$.

Quiz number 5 Solutions

Show all work. How you get your answer is just as important, if not more important, than the answer itself. If you think it, write it!

13. Find bases for the column space of the matrix $A = \begin{pmatrix} 2 & 3 & 4 \\ 5 & 7 & 9 \\ 1 & 4 & 7 \end{pmatrix}$, by

(a) row reducing the matrix A ,

(b) row reducing the transpose A^T of the matrix A .

$$A = \begin{pmatrix} 2 & 3 & 4 \\ 5 & 7 & 9 \\ 1 & 4 & 7 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 4 & 7 \\ 2 & 3 & 4 \\ 1 & 4 & 7 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 4 & 7 \\ 0 & -5 & -10 \\ 0 & -13 & -26 \end{pmatrix}$$

$$\rightarrow \begin{pmatrix} 1 & 4 & 7 \\ 0 & 1 & 2 \\ 0 & -13 & -26 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 4 & 7 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{pmatrix}$$

The REF has pivots in the first two columns, so the first two columns of A form a basis for $\text{col}(A)$.

$$\text{basis} = \begin{bmatrix} 2 \\ 5 \\ 1 \end{bmatrix} , \begin{bmatrix} 3 \\ 7 \\ 4 \end{bmatrix}$$

$$A^T = \begin{pmatrix} 2 & 5 & 1 \\ 3 & 7 & 4 \\ 4 & 8 & 7 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 5/2 & 1/2 \\ 3 & 7 & 4 \\ 4 & 8 & 7 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 5/2 & 1/2 \\ 0 & -1/2 & 5/2 \\ 0 & -1 & 5 \end{pmatrix}$$

$$\rightarrow \begin{pmatrix} 1 & 5/2 & 1/2 \\ 0 & 1 & -5 \\ 0 & -1 & 5 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 0 & 13 \\ 0 & 1 & -5 \\ 0 & 0 & 0 \end{pmatrix}$$

The non-zero rows, $[1 \ \ 0 \ \ 13]$ and $[0 \ \ 1 \ \ -5]$, form a basis for $\text{row}(A^T)$. So their transposes,

$$\begin{bmatrix} 1 \\ 0 \\ 13 \end{bmatrix} \text{ and } \begin{bmatrix} 0 \\ 1 \\ -5 \end{bmatrix} \text{ , form a basis for } \text{col}(A) \text{ .}$$

3. (15 pts.) Find the value of Ax closest to b, where

$$A = \begin{pmatrix} 1 & 2 \\ 1 & 2 \\ 0 & 1 \end{pmatrix} \qquad \text{and} \qquad b = \begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix}$$

$$A^T A \bar{x} = A^T b$$

$$A^T A = \begin{pmatrix} 1 & 1 & 0 \\ 2 & 1 & 1 \end{pmatrix}\begin{pmatrix} 1 & 2 \\ 1 & 1 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 2 & 3 \\ 3 & 6 \end{pmatrix} \qquad A^T b = \begin{pmatrix} 1 & 1 & 0 \\ 2 & 1 & 1 \end{pmatrix}\begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix}$$

$$\left(\begin{array}{cc|c} 2 & 3 & 3 \\ 3 & 6 & 6 \end{array}\right) \longrightarrow \left(\begin{array}{cc|c} 1 & 2 & 2 \\ 2 & 3 & 3 \end{array}\right) \qquad = \begin{pmatrix} 3 \\ 6 \end{pmatrix}$$

$$\longrightarrow \left(\begin{array}{cc|c} 1 & 2 & 2 \\ 0 & -1 & -1 \end{array}\right) \longrightarrow \left(\begin{array}{cc|c} 1 & 0 & 0 \\ 0 & 1 & 1 \end{array}\right) \qquad \begin{array}{l} x_1 = 0 \\ x_2 = 1 \end{array}$$

$$A\bar{x} = \begin{pmatrix} 1 & 2 \\ 1 & 1 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix}$$

$$(A^T A)^{-1} = \frac{1}{3}\begin{pmatrix} 6 & -3 \\ -3 & 2 \end{pmatrix} \qquad A(A^T A)^{-1} A^T$$

$$= \frac{1}{3}\begin{pmatrix} 1 & 2 \\ 1 & 1 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 6 & -3 \\ -3 & 2 \end{pmatrix}\begin{pmatrix} 1 & 1 & 0 \\ 2 & 1 & 1 \end{pmatrix}$$

$$\downarrow = \frac{1}{3}\begin{pmatrix} 1 & 2 \\ 1 & 1 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 6 & -3 \\ -3 & 2 \end{pmatrix}\begin{pmatrix} 1 & 1 & 0 \\ 2 & 1 & 1 \end{pmatrix} = \frac{1}{3}\begin{pmatrix} 1 & 2 \\ 1 & 1 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 0 & 3 & -3 \\ 1 & -1 & 2 \end{pmatrix}$$

$$= \frac{1}{3}\begin{pmatrix} 2 & 1 & 1 \\ 1 & 2 & -1 \\ 1 & -1 & 2 \end{pmatrix} \qquad (\quad)b = \frac{1}{3}\begin{pmatrix} 6 \\ 3 \\ 3 \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix}$$

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4. (~~15~~ pts.) Find the determinant of the matrix

$$A = \begin{pmatrix} 1 & 2 & -7 \\ 1 & 0 & -3 \\ 3 & 1 & 3 \end{pmatrix}$$

Based on this, find the determinants of the matrices $B = A^{-1}$, $C = A^T$, and $D = A^T A$. (Hint: you don't need to compute these matrices....)

$\det A = 1(0+3) - 1(6+7) + 3(-6-0)$

$\quad = 3 - 13 - 18 = 3 - 31 = -28$

$\det(B) = \dfrac{1}{\det A} = \dfrac{1}{-28}$

$\det(C) = \det A) = -28$

$\det(D) = \det(A^T) \det(A) = (-28)^2$

4

20

7. (15 pts.) Find the orthogonal complement of the subspace W of R^4 spanned by the vectors

$$\begin{pmatrix} 1 \\ -1 \\ 0 \\ 2 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} -1 \\ 1 \\ 1 \\ 2 \end{pmatrix}$$

$$A = \begin{pmatrix} 1 & -1 \\ -1 & 1 \\ 0 & 1 \\ 2 & 2 \end{pmatrix} \qquad W = C(A)$$

$$W^{\perp} = N(A^{T}) \qquad A^{T} = \begin{pmatrix} 1 & -1 & 0 & 2 \\ -1 & 1 & 1 & 2 \end{pmatrix} \longrightarrow$$

$$\begin{pmatrix} 1 & -1 & 0 & 2 \\ 0 & 0 & 1 & 4 \end{pmatrix}$$

$$\underset{\text{free}}{\uparrow} \qquad \uparrow$$

$x_1 = x_2 - 2x_4$

$x_3 = -4x_4$

$$\begin{matrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{matrix} = \begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \end{pmatrix} x_2 + \begin{pmatrix} -2 \\ 0 \\ -4 \\ 1 \end{pmatrix} x_4$$

basis for $W^{\perp}$.

7

Name:

Math 314 Matrix Theory
Final Exam

Show all work. Include all steps necessary to arrive at an answer unaided by a mechanical computational device. The steps you take to your answer are just as important, if not more important, than the answer itself. If you think it, write it!

1. (25 pts.) Find bases for the column space, row space, and nullspace of the matrix
$$A = \begin{bmatrix} 1 & -1 & 2 \\ 2 & 1 & 2 \\ 1 & 5 & -2 \end{bmatrix}.$$

$$\begin{pmatrix} 1 & -1 & 2 \\ 2 & 1 & 2 \\ 1 & 5 & -2 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & -1 & 2 \\ 0 & 3 & -2 \\ 0 & 6 & -4 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & -1 & 2 \\ 0 & 3 & -2 \\ 0 & 0 & 0 \end{pmatrix} \qquad \frac{1}{3}z\begin{pmatrix} -4 \\ 2 \\ 3 \end{pmatrix}$$

$$\rightarrow \begin{pmatrix} 1 & -1 & 2 \\ 0 & 1 & -2/3 \\ 0 & 0 & 0 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 0 & 4/3 \\ 0 & 1 & -2/3 \\ 0 & 0 & 0 \end{pmatrix} \qquad \begin{matrix} x + 4/3 z = 0 \\ y - 2/3 z = 0 \end{matrix} \qquad \begin{pmatrix} x \\ y \\ z \end{pmatrix} = z \begin{pmatrix} -4/3 \\ 2/3 \\ 1 \end{pmatrix}$$

pivot — free

$$\text{Col}(A): \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}, \begin{pmatrix} -1 \\ 1 \\ 5 \end{pmatrix} \qquad \text{row}(A): \begin{pmatrix} 1 \\ 0 \\ 4/3 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ -2/3 \end{pmatrix}$$

$$\text{null}(A): \begin{pmatrix} -4/3 \\ 2/3 \\ 1 \end{pmatrix}$$

1

2. (20 pts.) Find the inverse of the matrix $\quad A = \begin{bmatrix} 1 & -1 & 3 \\ 1 & 2 & 2 \\ 1 & 2 & 1 \end{bmatrix}$.

$$\left(\begin{array}{ccc|ccc} 1 & -1 & 3 & 1 & 0 & 0 \\ 1 & 2 & 2 & 0 & 1 & 0 \\ 1 & 2 & 1 & 0 & 0 & 1 \end{array} \right) \longrightarrow \left(\begin{array}{ccc|ccc} 1 & -1 & 3 & 1 & 0 & 0 \\ 0 & 3 & -1 & -1 & 1 & 0 \\ 0 & 3 & -2 & -1 & 0 & 1 \end{array} \right)$$

$$\longrightarrow \left(\begin{array}{ccc|ccc} 1 & -1 & 3 & 1 & 0 & 0 \\ 0 & 1 & -\frac{1}{3} & -\frac{1}{3} & \frac{1}{3} & 0 \\ 0 & 3 & -2 & -1 & 0 & 1 \end{array} \right) \longrightarrow \left(\begin{array}{ccc|ccc} 1 & -1 & 3 & 1 & 0 & 0 \\ 0 & 1 & -\frac{1}{3} & -\frac{1}{3} & \frac{1}{3} & 0 \\ 0 & 0 & -1 & 0 & -1 & 1 \end{array} \right)$$

$$\longrightarrow \left(\begin{array}{ccc|ccc} 1 & -1 & 0 & 1 & -3 & 3 \\ 0 & 1 & 0 & -\frac{1}{3} & \frac{2}{3} & -\frac{1}{3} \\ 0 & 0 & 1 & 0 & 1 & -1 \end{array} \right) \longrightarrow \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & \frac{2}{3} & -\frac{7}{3} & \frac{8}{3} \\ 0 & 1 & 0 & -\frac{1}{3} & \frac{2}{3} & -\frac{1}{3} \\ 0 & 0 & 1 & 0 & 1 & -1 \end{array} \right)$$

So $\quad A^{-1} = \begin{pmatrix} \frac{2}{3} & -\frac{7}{3} & \frac{8}{3} \\ -\frac{1}{3} & \frac{2}{3} & -\frac{1}{3} \\ 0 & 1 & -1 \end{pmatrix} = \frac{1}{3} \begin{pmatrix} 2 & -7 & 8 \\ -1 & 2 & -1 \\ 0 & 3 & -3 \end{pmatrix}$

2

3. (20 pts.) For which value(s) of x do the vectors

$$\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ x \\ 2 \end{bmatrix}, \begin{bmatrix} x \\ 2 \\ 2 \end{bmatrix} \qquad \textbf{span } \mathbf{R}^3 \text{ ?}$$

want: no inconsistent system $\Rightarrow$ no row of 0's.

$$\left(\begin{array}{ccc} 1 & -1 & x \\ 1 & x & 2 \\ 1 & 2 & 2 \end{array} \right) \rightarrow \left(\begin{array}{ccc} 1 & -1 & x \\ 0 & x+1 & 2-x \\ 0 & 3 & 2-x \end{array} \right) \rightarrow \left(\begin{array}{ccc} 1 & -1 & x \\ 0 & x-2 & 3x \cdot 0 \\ 0 & 3 & 2-x \end{array} \right)$$

$$\rightarrow \left(\begin{array}{ccc} 1 & -1 & x \\ 0 & 3 & 2-x \\ 0 & x-2 & 0 \end{array} \right) \rightarrow \left(\begin{array}{ccc} 1 & -1 & x \\ 0 & 1 & 0 \\ 0 & 3 & 2-x \end{array} \right) \rightarrow \left(\begin{array}{ccc} 1 & -1 & x \\ 0 & 1 & 0 \\ 0 & 0 & 2-x \end{array} \right)$$

unless $x=2$

$\Rightarrow$ 3 pivots unless $x=2$. When $x=2$,

is $\left(\begin{array}{ccc} 1 & -1 & 2 \\ 0 & 0 & 0 \\ 0 & 3 & 0 \end{array} \right)$

$\downarrow$

$\left(\begin{array}{ccc} 1 & -1 & 2 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{array} \right)$

So if $x \neq 2$, they span. If $x=2$, they don't.

or: compute $\det \left(\begin{array}{ccc} 1 & -1 & x \\ 1 & x & 2 \\ 1 & 2 & 2 \end{array} \right)$

$$= 1 \left| \begin{array}{cc} x & 2 \\ 2 & 2 \end{array} \right| - (-1) \left| \begin{array}{cc} 1 & 2 \\ 1 & 2 \end{array} \right| + x \left| \begin{array}{cc} 1 & x \\ 1 & 2 \end{array} \right|$$

$$= 1(2x-4) + 1(2-2) + x(2-x)$$

$$= 2x-4 + 2x-x^2 = -x^2 + 4x - 4 = -(x^2-4x+4) = -(x-2)^2$$

$\Rightarrow 0$ only for $x=2$

$x \neq 2 \Rightarrow \det \neq 0 \Rightarrow$ invertible $\Rightarrow$ columns span

$x=2 \Rightarrow \det = 0 \Rightarrow$ not invertible $\Rightarrow$ columns don't span.

4. (20 pts.) Use the Gram-Schmidt orthogonalization process to construct an orthogonal set from the (you may assume linearly independent) vectors

$$\vec{w}_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} , \vec{w}_2 = \begin{pmatrix} 1 \\ 1 \\ -1 \\ 1 \end{pmatrix} , \vec{w}_3 = \begin{pmatrix} 1 \\ 1 \\ -1 \\ -1 \end{pmatrix} .$$

$$V_1 = W_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}$$

$$V_2 = W_2 - proj_{V_1} W_2 = \begin{pmatrix} 1 \\ 1 \\ -1 \\ 1 \end{pmatrix} - \frac{W_2 \cdot V_1}{V_1 \cdot V_1} V_1$$

$$= \begin{pmatrix} 1 \\ 1 \\ -1 \\ 1 \end{pmatrix} - \frac{2}{4} \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ -1 \\ 1 \end{pmatrix} - \begin{pmatrix} \frac{1}{2} \\ \frac{1}{2} \\ \frac{1}{2} \\ \frac{1}{2} \end{pmatrix} = \begin{pmatrix} \frac{1}{2} \\ \frac{1}{2} \\ -\frac{3}{2} \\ \frac{1}{2} \end{pmatrix} V_2$$

$$V_3 = W_3 - proj_{V_1} W_3 - proj_{V_2} W_3$$

$$= \begin{pmatrix} 1 \\ 1 \\ -1 \\ -1 \end{pmatrix} - \frac{W_3 \cdot V_1}{V_1 \cdot V_1} V_1 - \frac{W_3 \cdot V_2}{V_2 \cdot V_2} V_2$$

$$= \begin{pmatrix} 1 \\ 1 \\ -1 \\ -1 \end{pmatrix} - \frac{0}{4} \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} - \frac{\frac{1}{2} + \frac{1}{2} + \frac{3}{2} - \frac{1}{2}}{\frac{1}{4} + \frac{1}{4} + \frac{9}{4} + \frac{1}{4}} \begin{pmatrix} \frac{1}{2} \\ \frac{1}{2} \\ -\frac{3}{2} \\ \frac{1}{2} \end{pmatrix}$$

$$= \begin{pmatrix} 1 \\ 1 \\ -1 \\ -1 \end{pmatrix} - \frac{2}{3} \begin{pmatrix} \frac{1}{2} \\ \frac{1}{2} \\ -\frac{3}{2} \\ \frac{1}{2} \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ -1 \\ -1 \end{pmatrix} - \begin{pmatrix} \frac{1}{3} \\ \frac{1}{3} \\ -1 \\ \frac{1}{3} \end{pmatrix} = \begin{pmatrix} \frac{2}{3} \\ \frac{2}{3} \\ 0 \\ -\frac{4}{3} \end{pmatrix}$$

$$\text{so } \perp \text{ set is } \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} \frac{1}{2} \\ \frac{1}{2} \\ -\frac{3}{2} \\ \frac{1}{2} \end{pmatrix}, \begin{pmatrix} \frac{2}{3} \\ \frac{2}{3} \\ 0 \\ -\frac{4}{3} \end{pmatrix}$$

or!

$$\begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ -3 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 0 \\ -2 \end{pmatrix}$$

5. (25 pts.) Find the orthogonal projection of the vector $\vec{v} = \begin{pmatrix} 3 \\ 2 \\ 1 \end{pmatrix}$ onto the column space of the matrix

$$A = \begin{bmatrix} 1 & -1 \\ 1 & 1 \\ 2 & 1 \end{bmatrix}.$$

Find $A\vec{x}$, where $A^T A \vec{x} = A^T \vec{v}$

$A^T A = \begin{bmatrix} 1 & 1 & 2 \\ -1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & -1 \\ 1 & 1 \\ 2 & 1 \end{bmatrix} = \begin{bmatrix} 6 & 2 \\ 2 & 3 \end{bmatrix}$

$A^T \vec{v} = \begin{bmatrix} 1 & 1 & 2 \\ -1 & 1 & 1 \end{bmatrix} \begin{pmatrix} 3 \\ 2 \\ 1 \end{pmatrix} = \begin{pmatrix} 7 \\ 0 \end{pmatrix}$

$\left(\begin{array}{cc|c} 6 & 2 & 7 \\ 2 & 3 & 0 \end{array} \right) \rightarrow \left(\begin{array}{cc|c} 1 & 3/2 & 0 \\ 6 & 2 & 7 \end{array} \right) \rightarrow \left(\begin{array}{cc|c} 1 & 3/2 & 0 \\ 0 & -7 & 7 \end{array} \right)$

$\rightarrow \left(\begin{array}{cc|c} 1 & 3/2 & 0 \\ 0 & 1 & -1 \end{array} \right) \rightarrow \left(\begin{array}{cc|c} 1 & 0 & 3/2 \\ 0 & 1 & -1 \end{array} \right)$ so $\vec{x} = \begin{pmatrix} 3/2 \\ -1 \end{pmatrix}$

$\text{proj}_{\text{col}(A)} \vec{v} = A\vec{x} = \begin{pmatrix} 1 & -1 \\ 1 & 1 \\ 2 & 1 \end{pmatrix} \begin{pmatrix} 3/2 \\ -1 \end{pmatrix} = \begin{pmatrix} 5/2 \\ 1/2 \\ 2 \end{pmatrix} = \frac{1}{2} \begin{pmatrix} 5 \\ 1 \\ 4 \end{pmatrix}$
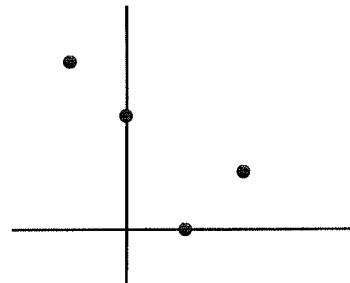
Check! $\vec{v} - A\vec{x} = \begin{pmatrix} 1/2 \\ 3/2 \\ -1 \end{pmatrix}$ is $\perp \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix}, \begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix}$ ✓

5

6. (20 pts.) Find the line $y = ax + b$ that gives the least squares best fit to the data points

$$(-1, 3) , (0, 2) , (1, 0) , (2, 1) .$$

$y = ax + b$ is best fit if $\begin{pmatrix} a \\ b \end{pmatrix}$ is

closest soln to

$$A\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} -1 & 1 \\ 0 & 1 \\ 1 & 1 \\ 2 & 1 \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 3 \\ 2 \\ 0 \\ 1 \end{pmatrix} = \vec{v}$$

solve $A^T A \begin{pmatrix} a \\ b \end{pmatrix} = A^T \vec{v}$

$$A^T A = \begin{pmatrix} -1 & 0 & 1 & 2 \\ 1 & 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} -1 & 1 \\ 0 & 1 \\ 1 & 1 \\ 2 & 1 \end{pmatrix}$$

$$A^T \vec{v} = \begin{pmatrix} -1 & 0 & 1 & 2 \\ 1 & 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} 3 \\ 2 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} -1 \\ 6 \end{pmatrix}$$

$$= \begin{pmatrix} 6 & 2 \\ 2 & 4 \end{pmatrix}$$

$$\begin{pmatrix} 6 & 2 & | & -1 \\ 2 & 4 & | & 6 \end{pmatrix} \to \begin{pmatrix} 2 & 4 & | & 6 \\ 6 & 2 & | & -1 \end{pmatrix} \to \begin{pmatrix} 1 & 2 & | & 3 \\ 6 & 2 & | & -1 \end{pmatrix} \to \begin{pmatrix} 1 & 2 & | & 3 \\ 0 & -10 & | & -19 \end{pmatrix}$$

$$\to \begin{pmatrix} 1 & 2 & | & 3 \\ 0 & 1 & | & \frac{19}{10} \end{pmatrix} \to \begin{pmatrix} 1 & 0 & | & -\frac{8}{10} \\ 0 & 1 & | & \frac{19}{10} \end{pmatrix}$$

$a = -\frac{8}{10} = -\frac{4}{5}$

$b = \frac{19}{10}$

$y = -\frac{4}{5}x + \frac{19}{10}$ is the least squares best fit.

$-\frac{4}{5}x + \frac{19}{10}$

7. (20 pts.) Find the eigenvalues, and associated eigenbases, for the matrix
$$A = \begin{pmatrix} 0 & 2 \\ 3 & 1 \end{pmatrix}.$$

$$\det(A - \lambda I) = \begin{vmatrix} -\lambda & 2 \\ 3 & 1-\lambda \end{vmatrix} = (-\lambda)(1+\lambda) - 6$$

$$= \lambda^2 - \lambda + 6 = (\lambda - 3)(\lambda + 2)$$

$\lambda = 3, -2$

$\lambda = 3$: $A - 3I = \begin{pmatrix} -3 & 2 \\ 3 & -2 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & -2/3 \\ 0 & 0 \end{pmatrix}$

$x - \frac{2}{3}y = 0 \qquad x = \frac{2}{3} \qquad \begin{pmatrix} x \\ y \end{pmatrix} = y \begin{pmatrix} 2/3 \\ 1 \end{pmatrix}$

$\lambda = -2 \qquad A - (-2I) = A + 2I = \begin{pmatrix} 2 & 2 \\ 3 & 2 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 1 \\ 3 & 3 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}$

$x + y = 0 \qquad x = -y \qquad \begin{pmatrix} x \\ y \end{pmatrix} = y \begin{pmatrix} -1 \\ 1 \end{pmatrix}$

e-value $\lambda = 3$: e-basis is $\begin{pmatrix} 2/3 \\ 1 \end{pmatrix}$

e-value $\lambda = -2$ e-basis is $\begin{pmatrix} -1 \\ 1 \end{pmatrix}$