One calculation to invert **ALL 3×3 matrices (that are invertible!) [TYPO FIXED]**

To invert an arbitrary 3×3 matrix $A = \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix}$, we use the super-augmented matrix, and row reduce!

$$(A|I_3) = \left(\begin{array}{ccc|ccc} a & b & c & 1 & 0 & 0 \\ d & e & f & 0 & 1 & 0 \\ g & h & i & 0 & 0 & 1 \end{array}\right) \longrightarrow \left(\begin{array}{ccc|ccc} 1 & \frac{b}{a} & \frac{c}{a} & \frac{1}{a} & 0 & 0 \\ d & e & f & 0 & 1 & 0 \\ g & h & i & 0 & 0 & 1 \end{array}\right) \longrightarrow \left(\begin{array}{ccc|ccc} 1 & \frac{b}{a} & \frac{c}{a} & \frac{1}{a} & 0 & 0 \\ 0 & \frac{ae-bd}{a} & \frac{af-cd}{a} & \frac{-d}{a} & 1 & 0 \\ g & h & i & 0 & 0 & 1 \end{array}\right)$$

$$\longrightarrow \left(\begin{array}{ccc|ccc} 1 & \frac{b}{a} & \frac{c}{a} & \frac{1}{a} & 0 & 0 \\ 0 & \frac{ae-bd}{a} & \frac{af-cd}{a} & \frac{-d}{a} & 1 & 0 \\ 0 & \frac{ah-bg}{a} & \frac{ai-cg}{a} & \frac{-g}{a} & 0 & 1 \end{array}\right) \longrightarrow \left(\begin{array}{ccc|ccc} 1 & \frac{b}{a} & \frac{c}{a} & \frac{1}{a} & 0 & 0 \\ 0 & 1 & \frac{af-cd}{ae-bd} & \frac{-d}{ae-bd} & \frac{a}{ae-bd} & 0 \\ 0 & \frac{ah-bg}{a} & \frac{ai-cg}{a} & \frac{-g}{a} & 0 & 1 \end{array}\right)$$

$$\longrightarrow \left(\begin{array}{ccc|ccc} 1 & \frac{b}{a} & \frac{c}{a} & \frac{1}{a} & 0 & 0 \\ 0 & 1 & \frac{af-cd}{ae-bd} & \frac{-d}{ae-bd} & \frac{a}{ae-bd} & 0 \\ 0 & 0 & \frac{ai-cg}{a} - \frac{(ah-bg)(af-cd)}{a(ae-bd)} & \frac{-g}{a} + \frac{(ah-bg)d}{a(ae-bd)} & \frac{-(ah-bg)}{ae-bd} & 1 \end{array}\right)$$

$$= \left(\begin{array}{ccc|ccc} 1 & \frac{b}{a} & \frac{c}{a} & \frac{1}{a} & 0 & 0 \\ 0 & 1 & \frac{af-cd}{ae-bd} & \frac{-d}{ae-bd} & \frac{a}{ae-bd} & 0 \\ 0 & 0 & \frac{\Delta}{a(ae-bd)} & \frac{dh-eg}{ae-bd} & \frac{-(ah-bg)}{ae-bd} & 1 \end{array}\right) = \left(\begin{array}{ccc|ccc} 1 & \frac{b}{a} & \frac{c}{a} & \frac{1}{a} & 0 & 0 \\ 0 & 1 & \frac{af-cd}{ae-bd} & \frac{-d}{ae-bd} & \frac{a}{ae-bd} & 0 \\ 0 & 0 & \frac{\Delta'}{(ae-bd)} & \frac{dh-eg}{ae-bd} & \frac{-(ah-bg)}{ae-bd} & 1 \end{array}\right)$$

where $\Delta = (ai-cg)(ae-bd) - (ah-bg)(af-cd) = a(aei - ceg - bdi - afh + bfg + cdh)$
$= a[a(ei-fh) - b(di-fg) + c(dh-eg)] = a\Delta'$

$$\longrightarrow \left(\begin{array}{ccc|ccc} 1 & \frac{b}{a} & \frac{c}{a} & \frac{1}{a} & 0 & 0 \\ 0 & 1 & \frac{af-cd}{ae-bd} & \frac{-d}{ae-bd} & \frac{a}{ae-bd} & 0 \\ 0 & 0 & 1 & \frac{(dh-eg)}{\Delta'} & \frac{-(ah-bg)}{\Delta'} & \frac{(ae-bd)}{\Delta'} \end{array}\right)$$

$$\longrightarrow \left(\begin{array}{ccc|ccc} 1 & \frac{b}{a} & \frac{c}{a} & \frac{1}{a} & 0 & 0 \\ 0 & 1 & 0 & \frac{-d}{ae-bd} - \frac{(af-cd)(dh-eg)}{(ae-bd)\Delta'} & \frac{a}{ae-bd} + \frac{(af-cd)(ah-bg)}{(ae-bd)\Delta'} & \frac{-(af-cd)}{\Delta'} \\ 0 & 0 & 1 & \frac{(dh-eg)}{\Delta'} & \frac{-(ah-bg)}{\Delta'} & \frac{(ae-bd)}{\Delta'} \end{array}\right)$$

$$= \left(\begin{array}{ccc|ccc} 1 & \frac{b}{a} & \frac{c}{a} & \frac{1}{a} & 0 & 0 \\ 0 & 1 & 0 & \frac{-d\Delta' - (af-cd)(dh-eg)}{(ae-bd)\Delta'} & \frac{a\Delta' + (af-cd)(ah-bg)}{(ae-bd)\Delta'} & \frac{-(af-cd)}{\Delta'} \\ 0 & 0 & 1 & \frac{dh-eg}{\Delta'} & \frac{-(ah-bg)}{\Delta'} & \frac{ae-bd}{\Delta'} \end{array}\right)$$

$$= \left(\begin{array}{ccc|ccc} 1 & \frac{b}{a} & \frac{c}{a} & \frac{1}{a} & 0 & 0 \\ 0 & 1 & 0 & \frac{-(ae-bd)(di-fg)}{(ae-bd)\Delta'} & \frac{(ae-bd)(ai-cg)}{(ae-bd)\Delta'} & \frac{-(af-cd)}{\Delta'} \\ 0 & 0 & 1 & \frac{(dh-eg)}{\Delta'} & \frac{-(ah-bg)}{\Delta'} & \frac{ae-bd}{\Delta'} \end{array}\right) \quad \text{(check it!)}$$

$$= \left(\begin{array}{ccc|ccc} 1 & \frac{b}{a} & \frac{c}{a} & \frac{1}{a} & 0 & 0 \\ 0 & 1 & 0 & \frac{-(di-fg)}{\Delta'} & \frac{ai-cg}{\Delta'} & \frac{-(af-cd)}{\Delta'} \\ 0 & 0 & 1 & \frac{dh-eg}{\Delta'} & \frac{-(ah-bg)}{\Delta'} & \frac{ae-bd}{\Delta'} \end{array}\right) \longrightarrow \left(\begin{array}{ccc|ccc} 1 & \frac{b}{a} & 0 & \frac{1}{a} - \frac{c(dh-eg)}{a\Delta'} & \frac{c(ah-bg)}{a\Delta'} & \frac{-c(ae-bd)}{a\Delta'} \\ 0 & 1 & 0 & \frac{-(di-fg)}{\Delta'} & \frac{ai-cg}{\Delta'} & \frac{-(af-cd)}{\Delta'} \\ 0 & 0 & 1 & \frac{dh-eg}{\Delta'} & \frac{-(ah-bg)}{\Delta'} & \frac{ae-bd}{\Delta'} \end{array}\right)$$

$$\longrightarrow \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & \frac{\Delta' - c(dh-eg) + b(di-fg)}{a\Delta'} & \frac{c(ah-bg) - b(ai-cg)}{a\Delta'} & \frac{-c(ae-bd) + b(af-cd)}{a\Delta'} \\ 0 & 1 & 0 & \frac{-(di-fg)}{\Delta'} & \frac{ai-cg}{\Delta'} & \frac{-(af-cd)}{\Delta'} \\ 0 & 0 & 1 & \frac{dh-eg}{\Delta'} & \frac{-(ah-bg)}{\Delta'} & \frac{ae-bd}{\Delta'} \end{array}\right)$$

$$= \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & \frac{a(ei-fh)}{a\Delta'} & \frac{-a(bi-ch)}{a\Delta'} & \frac{a(bf-ce)}{a\Delta'} \\ 0 & 1 & 0 & \frac{-(di-fg)}{\Delta'} & \frac{ai-cg}{\Delta'} & \frac{-(af-cd)}{\Delta'} \\ 0 & 0 & 1 & \frac{dh-eg}{\Delta'} & \frac{-(ah-bg)}{\Delta'} & \frac{ae-bd}{\Delta'} \end{array}\right) \quad \text{(check it!)} \quad = \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & \frac{ei-fh}{\Delta'} & \frac{-(bi-ch)}{\Delta'} & \frac{bf-ce}{\Delta'} \\ 0 & 1 & 0 & \frac{-(di-fg)}{\Delta'} & \frac{ai-cg}{\Delta'} & \frac{-(af-cd)}{\Delta'} \\ 0 & 0 & 1 & \frac{dh-eg}{\Delta'} & \frac{-(ah-bg)}{\Delta'} & \frac{ae-bd}{\Delta'} \end{array}\right)$$

$$\text{And so: } \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix}^{-1} = \begin{pmatrix} \frac{ei-fh}{\Delta'} & \frac{-(bi-ch)}{\Delta'} & \frac{bf-ce}{\Delta'} \\ \frac{-(di-fg)}{\Delta'} & \frac{ai-cg}{\Delta'} & \frac{-(af-cd)}{\Delta'} \\ \frac{dh-eg}{\Delta'} & \frac{-(ah-bg)}{\Delta'} & \frac{ae-bd}{\Delta'} \end{pmatrix} \quad !$$

[In particular, A is invertible precisely when $\Delta' = a(ei-fh) - b(di-fg) + c(dh-eg)$ is not zero.]