The Fibonacci sequence

The Fibonacci sequence is perhaps the most famous sequence of numbers in history; it starts as

$$0, 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144, 233, \ldots$$

We will call them F_i, with $F_0 = 0$ and $F_1 = 1$. The basic pattern in the sequence is $F_{n+1} = F_n + F_{n-1}$; each succeeding term is the sum of the previous two. The sequence originated with a thought experiment on the breeding of rabbits; the idea was that a rabbit can give birth when mature, after one year, and every year after, and so the population growth $(F_{n+1} - F_n)$ is equal to the number of mature rabbits (F_{n-1}). It has since been found in an immmense number of situations, so much so that there is an entire mathematical journal devoted to the study of this sequence and the ideas that come from it.

But we can find a <u>formula</u> for F_n by turning this into a matrix multiplication problem! If we keep track of two terms of the sequence at a time, this is enough to determine the next term; in matrix terms,

$$\begin{pmatrix} F_n \\ F_{n+1} \end{pmatrix} = \begin{pmatrix} F_n \\ F_{n-1} + F_n \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} F_{n-1} \\ F_n \end{pmatrix}$$

and so $\begin{pmatrix} F_n \\ F_{n+1} \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} F_{n-1} \\ F_n \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} F_{n-2} \\ F_{n-1} \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}^2 \begin{pmatrix} F_{n-2} \\ F_{n-1} \end{pmatrix}$

and repeating this argument yields $\begin{pmatrix} F_n \\ F_{n+1} \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}^n \begin{pmatrix} F_0 \\ F_1 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}^n \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

So "all" the we need in order to have a formula for F_n is to know what $\begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}^n$ is! But we can do this, by diagonalizing:

$\chi_A(t) = \det \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix} = (-t)(1-t) - 1 = t^2 - t - 1$, which by the Quadratic Formula has roots $t = \dfrac{1 \pm \sqrt{(-1)^2 - 4(1)(-1)}}{2} = \dfrac{1 \pm \sqrt{5}}{2}$. This suddenly doesn't look quite as much fun anymore, but we can find the corresponding eigenbases:

$$A - \frac{1+\sqrt{5}}{2} I = \begin{pmatrix} -\frac{1+\sqrt{5}}{2} & 1 \\ 1 & 1 - \frac{1+\sqrt{5}}{2} \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & \frac{1-\sqrt{5}}{2} \\ -\frac{1+\sqrt{5}}{2} & 1 \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & \frac{1-\sqrt{5}}{2} \\ 0 & 0 \end{pmatrix}$$

and so our e-vectors satisfy $x + \dfrac{1-\sqrt{5}}{2} y = 0$, so $\begin{pmatrix} \frac{\sqrt{5}-1}{2} \\ 1 \end{pmatrix}$ is a $\dfrac{1+\sqrt{5}}{2}$-eigenvector.

An analogous computation shows that $\begin{pmatrix} -\frac{1+\sqrt{5}}{2} \\ 1 \end{pmatrix}$ is a $\dfrac{1-\sqrt{5}}{2}$-eigenvector.

This means that, as we have seen in class,

$$\begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} \frac{\sqrt{5}-1}{2} & -\frac{1+\sqrt{5}}{2} \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} \frac{\sqrt{5}-1}{2} & -\frac{1+\sqrt{5}}{2} \\ 1 & 1 \end{pmatrix} \begin{pmatrix} \frac{1+\sqrt{5}}{2} & 0 \\ 0 & \frac{1-\sqrt{5}}{2} \end{pmatrix}, \text{ and so}$$

$$\begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} \frac{\sqrt{5}-1}{2} & -\frac{1+\sqrt{5}}{2} \\ 1 & 1 \end{pmatrix} \begin{pmatrix} \frac{1+\sqrt{5}}{2} & 0 \\ 0 & \frac{1-\sqrt{5}}{2} \end{pmatrix} \begin{pmatrix} \frac{\sqrt{5}-1}{2} & -\frac{1+\sqrt{5}}{2} \\ 1 & 1 \end{pmatrix}^{-1}$$

and <u>so</u>

$$\begin{aligned} \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}^n &= \begin{pmatrix} \frac{\sqrt{5}-1}{2} & -\frac{1+\sqrt{5}}{2} \\ 1 & 1 \end{pmatrix} \begin{pmatrix} \frac{1+\sqrt{5}}{2} & 0 \\ 0 & \frac{1-\sqrt{5}}{2} \end{pmatrix}^n \begin{pmatrix} \frac{\sqrt{5}-1}{2} & -\frac{1+\sqrt{5}}{2} \\ 1 & 1 \end{pmatrix}^{-1} \\ &= \begin{pmatrix} \frac{\sqrt{5}-1}{2} & -\frac{1+\sqrt{5}}{2} \\ 1 & 1 \end{pmatrix} \begin{pmatrix} (\frac{1+\sqrt{5}}{2})^n & 0 \\ 0 & (\frac{1-\sqrt{5}}{2})^n \end{pmatrix} \begin{pmatrix} \frac{\sqrt{5}-1}{2} & -\frac{1+\sqrt{5}}{2} \\ 1 & 1 \end{pmatrix}^{-1} \end{aligned}$$

and so!

$$\begin{aligned} \begin{pmatrix} F_n \\ F_{n+1} \end{pmatrix} &= \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}^n \begin{pmatrix} 0 \\ 1 \end{pmatrix} \\ &= \begin{pmatrix} \frac{\sqrt{5}-1}{2} & -\frac{1+\sqrt{5}}{2} \\ 1 & 1 \end{pmatrix} \begin{pmatrix} (\frac{1+\sqrt{5}}{2})^n & 0 \\ 0 & (\frac{1-\sqrt{5}}{2})^n \end{pmatrix} \begin{pmatrix} \frac{\sqrt{5}-1}{2} & -\frac{1+\sqrt{5}}{2} \\ 1 & 1 \end{pmatrix}^{-1} \begin{pmatrix} 0 \\ 1 \end{pmatrix} \end{aligned}$$

Computing the inverse, multiplying out, grabbing the first entry, and cleaning the resulting formula up a bit, this yields the formula

$$F_n = \frac{1}{\sqrt{5}}[(\frac{1+\sqrt{5}}{2})^n - (\frac{1-\sqrt{5}}{2})^n]$$

(It helps in the cleaning up process to note that $\frac{\sqrt{5}+1}{2} \cdot \frac{\sqrt{5}-1}{2} = 1$ ....)

As a practical matter, it is also worth noting that $| \frac{1-\sqrt{5}}{2} | < 0.7$, and so $(\frac{1-\sqrt{5}}{2})^n$ tends to 0 very quickly. So, for $n \geq 2$,

$$F_n \text{ is the integer closest to } \frac{1}{\sqrt{5}} \cdot (\frac{1+\sqrt{5}}{2})^n \ .$$

Also, since the term $(\frac{1-\sqrt{5}}{2})^n$ alternates sign, the nearest integer is alternately above and below $\frac{1}{\sqrt{5}} \cdot (\frac{1+\sqrt{5}}{2})^n$.